ISRAEL JOURNAL OF MATHEMATICS 120 (2000), 511–561

FULL MULTIPLICATIVITY OF GAMMA FACTORS FOR $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$

BY

DAVID SOUDRY

School of Mathematical Sciences, Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
e-mail: soudry@math.tau.ac.il

ABSTRACT

In this paper we prove the full multiplicativity (in both variables) of gamma factors for generic representations of $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$. These gamma factors are initially defined as proportionality factors of local functional equations, derived from a corresponding global theory of certain Rankin–Selberg integrals which interpolate standard L-functions for $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$.

0. Introduction, preliminaries and notation

In [S1,2] we defined local gamma factors $\gamma(\pi \times \tau, s, \psi)$ for a pair of generic representations π and τ of $\mathrm{SO}_{2\ell+1}(F)$ and $\mathrm{GL}_n(F)$ respectively, over a local field F. Here s is a complex variable and ψ is a nontrivial additive character of F. Our main task in this paper is to prove that the gamma factor is multiplicative in the first variable, when F is nonarchimedean. Namely, if π is induced from a maximal parabolic subgroup, with Levi part isomorphic to $\mathrm{GL}_k(F) \times \mathrm{SO}_{2\ell'+1}(F)$ $(k+\ell' = \ell)$, and from generic representations σ and π' of $\mathrm{GL}_k(F)$ and $\mathrm{SO}_{2\ell'+1}(F)$ respectively, then

THEOREM 1:

$$\gamma(\pi \times \tau, s, \psi) = \omega_\tau(-1)^k \gamma(\sigma \times \tau, s, \psi)\gamma(\hat{\sigma} \times \tau, s, \psi)\gamma(\pi' \times \tau, s, \psi). \tag{0.1}$$

The first two gamma factors are ones for $\mathrm{GL}_k \times \mathrm{GL}_n$ (see [J.PS.S]). These gamma factors are identical to the corresponding local coefficients for $\mathrm{GL}_k \times \mathrm{GL}_n$,

Received October 21, 1998

defined by Shahidi (we use this fact in the paper). This was proved by Shahidi in [Sh3]. We proved (0.1) in case F is archimedean (see [S2]) and in case F is nonarchimedean and $\ell < n$ (see [S1]). To complete the proof of (0.1) in case $\ell \geq n$, we prove a partial multiplicativity of the gamma factor in the *second variable*. More precisely, assume that

$$\tau = \mathrm{Ind}_{P'_{1,n-1}}^{\mathrm{GL}_n(F)} \mu \otimes \tau' \quad \text{(normalized induction)}$$

where $P'_{1,n-1}$ is the standard parabolic subgroup of $\mathrm{GL}_n(F)$ of type $(1, n-1)$, μ is a quasi-character of F^* and τ' is a generic representation of $\mathrm{GL}_{n-1}(F)$. We assume, for simplicity of future calculations, that $\mu(-1) = 1$. Then

THEOREM 2:

$$\gamma(\pi \times \tau, s, \psi) = \gamma(\pi \times \mu, s, \psi)\gamma(\pi \times \tau', s, \psi). \tag{0.2}$$

Using global arguments, we will conclude from Theorems 1 and 2 the multiplicativity of the gamma factor in the second variable as well, and this will conclude the full multiplicativity of the gamma factor.

THEOREM 3: *Assume that τ is induced from a maximal parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$, and from the irreducible (generic) representation $\tau_1 \otimes \tau_2$. Then*

$$\gamma(\pi \otimes \tau, s, \psi) = \gamma(\pi \otimes \tau_1, s, \psi)\gamma(\pi \otimes \tau_2, s, \psi).$$

These multiplicativity properties show that our gamma factor is identical with the Shahidi local coefficient on $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$. The multiplicativity of the Shahidi local coefficient is immediate from its definition and a similar property of intertwining operators, while the proof of this property of our gamma factor is long and very technical. However, our gamma factors appear in the local theory of Rankin–Selberg convolutions for $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$, which can locate poles of the corresponding tensor L-functions which, in turn, play an important role in the application of the converse theorem to the proof of existence of a lifting of cuspidal generic representations of $\mathrm{SO}_{2\ell+1}(\mathbb{A})$ to automorphic representations of $\mathrm{GL}_{2\ell}(\mathbb{A})$.

Let us explain how (0.2) and (0.1), for $\ell < n$, imply (0.1) for $\ell \geq n$. Assume that $\ell \geq n$ and π is induced from $\sigma \otimes \pi'$ as before. Take t, such that $n + t > \ell$, and choose characters $\mu_1, \ldots, \mu_t$ of F^* such that $\mu_i(-1) = 1, i = 1, \ldots, t$. Define

$$\tilde{\tau} = \mathrm{Ind}_{P'_{1,\ldots,1,n}}^{\mathrm{GL}_{n+t}(F)} \mu_1 \otimes \cdots \otimes \mu_t \otimes \tau.$$

$P'_{1,\ldots,1,n}$ is the standard parabolic subgroup of $\mathrm{GL}_{n+t}(F)$ of type $(1,\ldots,1,n)$. By (0.1) (for $\ell < n+t$),

$$\gamma(\pi \times \widetilde{\tau}, s, \psi) = \omega_{\widetilde{\tau}}(-1)^k \gamma(\sigma \times \widetilde{\tau}, s, \psi)\gamma(\hat{\sigma} \times \widetilde{\tau}, s, \psi)\gamma(\pi' \times \widetilde{\tau}, s, \psi). \tag{0.3}$$

A repeated application of (0.2) yields

$$\gamma(\pi' \times \widetilde{\tau}, s, \psi) = \Big[\prod_{i=1}^{t} \gamma(\pi' \times \mu_i, s, \psi)\Big] \cdot \gamma(\pi' \times \tau, s, \psi). \tag{0.4}$$

Also the gamma factors for $\mathrm{GL}_k \times \mathrm{GL}_{k'}$ are known to be multiplicative [J.PS.S], and so

$$\begin{aligned}&\gamma(\sigma \times \widetilde{\tau}, s, \psi)\gamma(\hat{\sigma} \times \widetilde{\tau}, s, \psi) = \\ &\Big[\prod_{i=1}^{t} \gamma(\sigma \times \mu_i, s, \psi)\gamma(\hat{\sigma} \times \mu_i, s, \psi)\Big]\gamma(\sigma \times \tau, s, \psi)\gamma(\hat{\sigma} \times \tau, s, \psi).\end{aligned} \tag{0.5}$$

Substitute (0.4), (0.5) in (0.3); then

$$\begin{aligned}\gamma(\pi \times \widetilde{\tau}, s, \psi) =& \omega_{\widetilde{\tau}}(-1)^k \gamma(\sigma \times \tau, s, \psi)\gamma(\hat{\sigma} \times \tau, s, \psi)\gamma(\pi' \times \tau, s, \psi) \cdot \\ &\cdot \prod_{i=1}^{t} \gamma(\sigma \times \mu_i, s, \psi)\gamma(\hat{\sigma} \times \mu_i, s, \psi)\gamma(\pi' \times \mu_i, s, \psi).\end{aligned} \tag{0.6}$$

A repeated application of (0.2) gives

$$\gamma(\pi \times \widetilde{\tau}, s, \psi) = \Big[\prod_{i=1}^{t} \gamma(\pi \times \mu_i, s, \psi)\Big]\gamma(\pi \times \tau, s, \psi). \tag{0.7}$$

LEMMA: *For a quasi-character μ of F^*, such that $\mu(-1) = 1$, $\gamma(\pi \times \mu, s, \psi)$ is multiplicative in π.*

Proof: Let $m > \ell$ and let

$$\tau_{m,\mu} = \mathrm{Ind}_B^{\mathrm{GL}_m(F)} \mu \otimes \cdots \otimes \mu,$$

where B is the Borel subgroup of $\mathrm{GL}_m(F)$. By (0.6),

$$\gamma(\pi \times \tau_{m,\mu}, s, \psi) = [\gamma(\sigma \times \mu, s, \psi)\gamma(\hat{\sigma} \times \mu, s, \psi)\gamma(\pi' \times \mu, s, \psi)]^m,$$

and by (0.7),

$$\gamma(\pi \times \tau_{m,\mu}, s, \psi) = [\gamma(\pi \times \mu, s, \psi)]^m.$$

Thus

$$[\gamma(\pi \times \mu, s, \psi)]^m = [\gamma(\sigma \times \mu, s, \psi)\gamma(\hat{\sigma} \times \mu, s, \psi)\gamma(\pi' \times \mu, s, \psi)]^m \tag{0.8}$$

for all $m > \ell$. This implies

$$\gamma(\pi \times \mu, s, \psi) = \gamma(\sigma \times \mu, s, \psi)\gamma(\hat{\sigma} \times \mu, s, \psi)\gamma(\pi' \times \mu, s, \psi). \tag{0.9}$$ ∎

Using (0.9), we can rewrite (0.7) as

$$\begin{aligned}\gamma(\pi \times \tilde{\tau}, s, \psi) = &\Big[\prod_{i=1}^{t} \gamma(\sigma \times \mu_i, s, \psi)\gamma(\hat{\sigma} \times \mu_i, s, \psi)\gamma(\pi' \times \mu_i, s, \psi)\Big]\cdot \\ &\cdot \gamma(\pi \times \tau, s, \psi).\end{aligned} \tag{0.10}$$

Now compare (0.10) with (0.6) to get

$$\gamma(\pi \times \tau, s, \psi) = \omega_\tau(-1)^k \sigma(\sigma \times \tau, s, \psi)\sigma(\hat{\sigma} \times \tau, s, \psi)\sigma(\pi' \times s, \psi).$$

This idea of proving (0.1) in case $\ell \geq n$, using (0.2) and (0.1) in case $\ell < n$, is similar to the one in [J.PS.S]. Most of the work of this paper is to prove Theorem 2.

Let us show how to prove Theorem 3, based on Theorems 1 and 2 and global arguments.

Proof of Theorem 3: Since Theorem 1 gives multiplicativity in the first variable, it is enough to prove Theorem 3 for supercuspidal π. Let

$$\tau = \mathrm{Ind}_{P_{n_1,\ldots,n_r}}^{\mathrm{GL}_n(F)} \tau_1 \otimes \cdots \otimes \tau_r$$

where $P_{n_1,\ldots,n_r}$ is the standard parabolic subgroup of $\mathrm{GL}_n(F)$ of type $(n_1, \ldots, n_r)$, $n_1 + \cdots + n_r = n$, and $\tau_1, \ldots, \tau_r$ are supercuspidal representations of $\mathrm{GL}_{n_1}(F), \ldots, \mathrm{GL}_{n_r}(F)$. It suffices to prove that

$$\gamma(\pi \otimes \tau, s, \psi) = \prod_{i=1}^{r} \gamma(\pi \otimes \tau_i, s, \psi).$$

We can embed π (resp. τ_i) as a local factor of an irreducible, automorphic, cuspidal generic representation $\tilde{\pi}$ (resp. $\tilde{\tau}_i$) of $\mathrm{SO}_{2\ell+1}(\mathbb{A})$ (resp. $\mathrm{GL}_{n_i}(\mathbb{A})$), where $\mathbb{A}$ is the ring of adeles of a number field k, such that at a certain place ν_0, $k_{\nu_0} = F$, $\tilde{\pi}_{\nu_0} = \pi$, $\tilde{\tau}_{i,\nu_0} = \tau_i$, and for all other finite places ν, $\tilde{\pi}_\nu$ and $\tilde{\tau}_{i,\nu}$ are unramified ($i = 1, \ldots, r$). See [Sh1, Sect. 4]. Assume first that $\ell < n$. The global Rankin–Selberg integrals for $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_n$ can be applied for $\tilde{\pi} \otimes \tilde{\tau}_z$ where

$z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ and $\widetilde{\tau}_z$ is the Eisenstein series on $GL_n(\mathbb{A})$ induced from $\widetilde{\tau}_1|\det\cdot|^{z_1} \otimes \cdots \otimes \widetilde{\tau}_r|\det\cdot|^{z_r}$. The Euler product expansion for the integrals is exactly the same as in the case we take a cusp form on $GL_n(\mathbb{A})$. The global functional equation of the Rankin–Selberg integrals implies that for $i < j$ and $\mathrm{Re}(z_i - z_j) \gg 0$,

$$\gamma(\widetilde{\pi}_\infty \otimes \widetilde{\tau}_{z,\infty}, s, \psi_\infty) \prod_{\nu<\infty} \gamma(\widetilde{\pi}_\nu \otimes \widetilde{\tau}_{z,\nu}, s, \psi_\nu) = 1. \tag{0.11}$$

(Here and below, we may interpret the infinite product as the finite product of local gamma factors over all places where not all data are unramified, times the quotient of the corresponding partial L functions at s and at $1-s$.) Here $\gamma(\widetilde{\pi}_\infty \otimes \widetilde{\tau}_{z,\infty}, s, \psi_\infty)$ is the product of $\gamma(\widetilde{\pi}_\nu \otimes \widetilde{\tau}_{z,\nu}, s, \psi_\nu)$ over all archimedean ν. Of course, we have, for $i = 1, \ldots, r$,

$$\gamma(\widetilde{\pi}_\infty \otimes \widetilde{\tau}_{i,\infty}, s+z_i, \psi_\infty)\gamma(\pi \otimes \tau_i, s+z_i, \psi) \prod_{\substack{\nu\neq\nu_0 \\ \nu<\infty}} \gamma(\widetilde{\pi}_\nu \otimes \widetilde{\tau}_{i,\nu}, s+z_i, \psi_\nu) = 1. \tag{0.12}$$

Since, for finite $\nu \neq \nu_0$, $\widetilde{\pi}_\nu$ and $\widetilde{\tau}_{i,\nu}$ are unramified, we have

$$\gamma(\widetilde{\pi}_\nu \otimes \widetilde{\tau}_{z,\nu}, s, \psi_\nu) = \prod_{i=1}^{r} \gamma(\widetilde{\pi}_\nu \otimes \widetilde{\tau}_{i,\nu}, s+z_i, \psi_\nu). \tag{0.13}$$

From [S2], we have

$$\gamma(\widetilde{\pi}_\infty \otimes \widetilde{\tau}_{z,\infty}, s, \psi_\infty) = \prod_{i=1}^{r} \gamma(\widetilde{\pi}_\infty \otimes \widetilde{\tau}_{i,\infty}, s+z_i, \psi_\infty). \tag{0.14}$$

We conclude from (0.11)–(0.14) that (for $\ell < n$)

$$\gamma(\pi \otimes \widetilde{\tau}_{z,\nu_0}, s, \psi) = \prod_{i=1}^{r} \gamma(\pi \otimes \tau_i, s+z_i, \psi). \tag{0.15}$$

It is clear that the l.h.s. of (0.15) is meromorphic in $(q^{-z_1}, \ldots, q^{-z_r}, q^{-s})$ and we can substitute $z = (0, \ldots, 0)$ to get

$$\gamma(\pi \otimes \tau, s, \psi) = \prod_{i=1}^{r} \gamma(\pi \otimes \tau_i, s, \psi). \tag{0.16}$$

Now assume that $\ell \geq n$. We repeat the trick we used before. Let $\mu_1, \ldots, \mu_t$ be, say, unramified characters of F^*, such that $n + t > \ell$, and consider, as before, $\tau' = \mathrm{Ind}_{P_{n_1,\ldots,n_r,1,\ldots,1}}^{GL_{n+t}(F)} \tau_1 \otimes \cdots \otimes \tau_r \otimes \mu_1 \otimes \cdots \otimes \mu_t$. Then by (0.16), we have

$$\gamma(\pi \otimes \tau', s, \psi) = \prod_{i=1}^{r} \gamma(\pi \otimes \tau_i, s, \psi) \prod_{i=1}^{t} \gamma(\pi \otimes \mu_i, s, \psi). \tag{0.17}$$

By Theorem 2,

$$\gamma(\pi \otimes \tau', s, \psi) = \gamma(\pi \otimes \tau, s, \psi) \prod_{i=1}^{t} \gamma(\pi \otimes \mu_i, s, \psi). \tag{0.18}$$

From (0.16) and (0.17) we conclude that

$$\gamma(\pi \otimes \tau, s, \psi) = \prod_{i=1}^{r} \gamma(\pi \otimes \tau_i, s, \psi) \tag{0.19}$$

for $\ell \geq n$ as well, and hence for all ℓ, n. This completes the proof of Theorem 3. ∎

The gamma factor is defined as a proportionality factor of a functional equation

$$\frac{\gamma(\pi \times \tau, s, \psi)}{\gamma(\tau, \Lambda^2, 2s-1, \psi)} A(W, \xi_{\tau,s}) = \widetilde{A}(W, \xi_{\tau,s}). \tag{0.20}$$

Here W is in the Whittaker model $W(\pi, \psi)$ of π with respect to ψ, $\xi_{\tau,s}$ is a section in $\rho_{\tau,s}$, the representation of $\mathrm{SO}_{2n}(F)$ (split) induced from the Siegel parabolic subgroup and the representation $\tau \otimes |\det\cdot|^{s-1/2}$ (normalized induction). A is a certain bilinear form and $\widetilde{A}$ is obtained from A by applying an intertwining operator to $\xi_{\tau,s}$. $\gamma(\tau, \Lambda^2, 2s-1, \psi)$ is the local coefficient of Shahidi [Sh2]. The precise definitions are recalled in Section 1. The proof of Theorem 2 is by directly proving (0.11) as *an identity* with $\gamma(\pi \times \tau, s, \psi)$ replaced by $\gamma(\pi \times \mu, s, \psi)\gamma(\pi \times \tau', s, \psi)$. The proof is long and very technical. It is in the same spirit as the other cases of multiplicativity mentioned before, but the calculations and specific tricks are different. For example, we have to use the multiplicativity of the Shahidi local coefficient. There are many places in the proof where we have to justify the passage from one local integral to another, after performing a formal manipulation. A typical justification consists of establishing a domain of absolute convergence of a multiple integral and also of a calculation of this integral for a special substitution. We will defer all these calculations to the last section of this paper. Finally, let us establish the main notation for this paper.

$F =$ local nonarchimedean field, with residue field of q elements, prime ideal $\mathcal{P}$ and ring of integers $\mathcal{O}$.

$$J_m = \begin{pmatrix} & & 1 \\ & 1 & \\ \cdot^{\cdot^{\cdot}} & & \\ 1 & & \end{pmatrix} \quad (m \times m \text{ matrix}).$$

$$\mathrm{SO}_m = \{ g \in \mathrm{SL}_m \mid {}^t g J_m g = J_m \}.$$

$G_\ell = \mathrm{SO}_{2\ell+1}(F)$.
$H_n = \mathrm{SO}_{2n}(F)$.

For $a \in \mathrm{GL}_\ell(F)$, denote $a^* = J_\ell {}^t a^{-1} J_\ell$ and $\hat{a} = \begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix}$. For a subgroup $B \subseteq \mathrm{GL}_\ell(F)$, let $\hat{B} = \{\hat{b} \mid b \in B\}$.

$A_\ell =$ diagonal subgroup of $\mathrm{GL}_\ell(F)$.
$Z_\ell =$ standard maximal unipotent subgroup of $\mathrm{GL}_\ell(F)$.
$N_\ell =$ standard maximal unipotent subgroup of G_ℓ. For a matrix $x \in M_{p\times q}(F)$, let $x' = -J_q {}^t x J_p$.
$Q_n =$ Siegel parabolic subgroup of H_n. Its Levi decomposition is

$$Q_n = L_n \ltimes U_n.$$

$L_n = \{m(a) = \begin{pmatrix} a & \\ & a^* \end{pmatrix} \mid a \in \mathrm{GL}_n(F)\}$.
$U_n = \{u(x) = \begin{pmatrix} I_n & x \\ & I_n \end{pmatrix} \mid x = x'\}$.
$\overline{U}_n = \{\overline{u}(x) = \begin{pmatrix} I_n & \\ x & I_n \end{pmatrix} \mid x = x'\}$.

For a subgroup $B \subset \mathrm{GL}_n(F)$, we denote $m(B) = \{m(b) \mid b \in \mathrm{GL}_n(F)\}$.
$V_n =$ standard maximal unipotent subgroup of H_n. $V_n = m(Z_n)U_n$.
$R_k =$ standard parabolic subgroup of H_n, which preserves a k-dimensional isotropic subspace. Levi decomposition: $R_k = M(R_k) \ltimes U(R_k)$ ($R_n = Q_n$).
$P'_{k,n-k} =$ standard parabolic subgroup of GL_n, of type $(k, n-k)$, $k = 1, \ldots, n-1$.

For $\ell < n$, we denote

$$r = n - \ell - 1$$

and $i_{\ell,n}$ denotes the embedding of G_ℓ in H_n given by

$$i_{\ell,n}(G_\ell) = \left\{ \begin{pmatrix} A & & B \\ & I_{2r} & \\ C & & D \end{pmatrix} \in H_n \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} e_0 = e_0 \right\},$$

where e_0 is the column vector in $F^{2\ell+2}$, with 1 at its $\ell+1$ coordinate, -1 at its $\ell+2$ coordinate and zero elsewhere. For $\ell \geq n$, $j_{n,\ell}$ denotes the embedding of H_n in G_ℓ given by

$$j_{n,\ell}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} A & & B \\ & I_{2(\ell-n)+1} & \\ C & & D \end{pmatrix}.$$

F^n = space of row vectors of dimension n over F.

F_n = space of column vectors of dimension n over F.

ψ = a nontrivial character of F. We let ψ denote the standard non-degenerate character it defines on Z_m, N_ℓ, V_n.

Given a representation π, which admits a unique Whittaker model with respect to a character θ, we denote this Whittaker model by $W(\pi,\theta)$. Induction of representations is assumed to be in normalized form. For a representation π, we denote by V_π a vector space realization of the action of π. If π has a central character, we denote it by ω_π.

1. Definition of $\gamma(\pi \times \tau, s, \psi)$

We recall, in this section, the definition of the gamma factor. Let π and τ be irreducible, generic representations of G_ℓ and $\mathrm{GL}_n(F)$ respectively. For $s \in \mathbb{C}$, let $\tau_s = \tau \otimes |\det\cdot|^{s-1/2}$, and consider $\rho_{\tau,s} = \mathrm{Ind}_{Q_n}^{H_n} \tau_s$. We realize τ in its Whittaker model $W(\tau,\psi^{-1})$. The elements of $V_{\rho_{\tau,s}}$ are smooth functions $\xi_{\tau,s}$ on H_n, which take values in $W(\tau,\psi^{-1})$, and regarding $\xi_{\tau,s}$ as a function on $H_n \times \mathrm{GL}_n(F)$,

$$\xi_{\tau,s}(m(a)u(b)h,x) = |\det a|^{s+(n-2)/2}\xi_{\tau,s}(h,xa), \quad h \in H_n,\ x \in \mathrm{GL}_n(F).$$

Put $f_{\xi_{\tau,s}}(h) = \xi_{\tau,s}(h,I_n)$. The integrals defined in [S1], for $W \in W(\pi,\psi)$ and $\xi_{\tau,s} \in V_{\rho_{\tau,s}}$, which are absolutely convergent in a right half plane and are rational functions in q^{-s}, are as follows.

CASE $\ell < n$:

$$A(W,\xi_{\tau,s}) = \int_{N_\ell\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}} f_{\xi_{\tau,s}}(\overline{x}\beta_{\ell,n}i_{\ell,n}(g))\psi_a(\overline{x})d\overline{x}dg.$$

Here

$$\beta_{\ell,n} = \begin{cases} \left(\begin{array}{c|cc|c} I_{\ell+1} & & & \\ \hline & & I_r & \\ & I_r & & \\ \hline & & & I_{\ell+1} \end{array}\right), & r = n-\ell-1 \text{ even}, \\[2ex] \left(\begin{array}{c|cccc|c} I_\ell & & & & & \\ \hline & & & & 1 & \\ & & & I_r & & \\ & & I_r & & & \\ & 1 & & & & \\ \hline & & & & & I_\ell \end{array}\right), & r \text{ odd}, \end{cases}$$

$$\overline{X}^{(\ell,n)} = \left\{ \overline{x} = \overline{u} \begin{pmatrix} v & z \\ 0 & v' \end{pmatrix} \in H_n \middle| v \in M_{r\times(\ell+1)}(F) \right\},$$
$$\psi_a(\overline{x}) = \psi(v_{r,\ell+1}).$$

CASE $\ell \geq n$:

$$A(W,\xi_{\tau,s}) = \int_{V_n\backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\overline{x} j_{n,\ell}(h)) f_{\xi_{\tau,s}}(h) d\overline{x} dh.$$

Here

$$\overline{X}_{(n,\ell)} = \left\{ \begin{pmatrix} I_n & \\ y & I_{\ell-n} \end{pmatrix}^{\wedge} \middle| y \in M_{(\ell-n)\times n}(F) \right\}.$$

Let

$$w_n = \begin{cases} \begin{pmatrix} & I_n \\ I_n & \end{pmatrix}, & n \text{ even} \\ \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} \begin{pmatrix} & & 1 \\ & I_{2n-2} & \\ 1 & & \end{pmatrix}, & n \text{ odd} \end{cases}$$

and consider the intertwining operator $M(w_n, \xi_{\tau,s})$ of $\rho_{\tau,s}$ corresponding to w_n. In [S1] we also consider $\widetilde{A}(W,\xi_{\tau,s})$ obtained (roughly) from $A(W,\xi_{\tau,s})$ by applying the intertwining operator to $\xi_{\tau,s}$. These are defined as follows

CASE $\ell < n, n$ even:

$$\widetilde{A}(W,\xi_{\tau,s}) = \int_{N_\ell\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} M(w_n,\xi_{\tau,s})(\overline{x}\beta_{\ell,n} i_{\ell,n}(g), b^*_{\ell,n}) \psi_a(\overline{x}) d\overline{x} dg.$$

Here $b_{\ell,n} = \mathrm{diag}(1,-1,1-1,\ldots,1,-1)$.

CASE $\ell < n, n$ odd:

$$\widetilde{A}(W,\xi_{\tau,s}) = \int_{N_\ell\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} \xi^*_{\tau^*,1-s}(\overline{x}\eta_{\ell,n} m(\varepsilon_{\ell,n}) i_{\ell,n}(g), I_n) \psi_a^{-1}(\overline{x}) d\overline{x} dg.$$

Here

$$\xi^*_{\tau^*,1-s}(h,c) = M(w_n,\xi_{\tau,s})(h^{\omega_n}, b^*_{\ell,n} c^*),$$

where

$$\omega_n = \begin{pmatrix} I_{n-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-1} \end{pmatrix}, \quad h^{\omega_n} = \omega_n^{-1} h \omega_n,$$

$$b_{\ell,n} = \mathrm{diag}(1,-1,1,-1,\ldots,-1,1) \cdot \begin{pmatrix} I_{\ell+1} & \\ & -I_r \end{pmatrix},$$

$$\varepsilon_{\ell,n} = \begin{cases} \left(\begin{array}{c|cc} I_{\ell+1} & & \\ \hline & & 1 \\ & I_{r-1} & \end{array}\right), & r \geq 2, \\ I_n, & r = 0, 1, \end{cases}$$

$$\eta_{\ell,n} = \begin{cases} \left(\begin{array}{c|cc|c} I_{\ell+2} & & & \\ \hline & & I_{r-1} & \\ & I_{r-1} & & \\ \hline & & & I_{\ell+2} \end{array}\right), & r \text{ odd}, r \geq 3, \\ \left(\begin{array}{c|c|ccc|c|c} I_\ell & & & & & & \\ \hline & & & & & 1 & \\ \hline & & 1 & & & & \\ & & & & I_{r-1} & & \\ & & & I_{r-1} & & & \\ & & & & 1 & & \\ \hline & 1 & & & & & \\ \hline & & & & & & I_\ell \end{array}\right), & r \text{ even}, r \geq 2, \\ I_{2n}, & r = 0, 1. \end{cases}$$

Case $\ell \geq n, n$ even:

$$\widetilde{A}(W, \xi_{\tau,s}) = \int_{V_n \backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\overline{x} j_{n,\ell}(h)) M(w_n, \xi_{\tau,s}) h, b_n^*) d\overline{x} dh.$$

Case $\ell \geq n, n$ odd:

$$\widetilde{A}(W, \xi_{\tau,s}) = \int_{V_n \backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\hat{c}_{n,\ell} \overline{x} j_{n,\ell}(h) a_{n,\ell}) M(w_n, \xi_{\tau,s})(h^{\omega_n}, b_n) d\overline{x} dh.$$

Here

$$a_{n,\ell} = \begin{pmatrix} I_\ell & & \\ & -1 & \\ & & I_\ell \end{pmatrix} j_{n,\ell}(\omega_n),$$
$$b_n = \mathrm{diag}(1, -1, 1, -1, \ldots),$$
$$c_{n,\ell} = \begin{pmatrix} I_n & \\ & -I_{\ell-n} \end{pmatrix}.$$

The functional equation asserts that there is a rational function in q^{-s}, $\Gamma(\pi \times \tau, s, \psi)$, such that

$$\Gamma(\pi \times \tau, s, \psi) A(W, \xi_{\tau,s}) = \widetilde{A}(W, \xi_{\tau,s}),$$

for all $W\in W(\pi,\psi)$ and all holomorphic sections $\xi_{\tau,s}$.

Let us specify the local coefficient. Consider the Whittaker model of $\rho_{\tau,s}$ given by the following Jacquet integrals:

(1.1)

$$W_{\xi_{\tau,s}}(h)=\begin{cases}\int \xi_{\tau,s}(w_n^{-1}u(x)h,I)\psi(x_{n-1,1})dx, & n \text{ even},\\ \int \xi_{\tau,s}\left(w_n^{-1}\begin{pmatrix} I_{n-1} & v & 0 & y\\ 0 & 1 & 0 & 0\\ & & 1 & v'\\ & & & I_{n-1}\end{pmatrix}h,I\right)\psi(v_{n-1})dvdy, & n \text{ odd},\end{cases}$$

In the first case, x varies over $\{e\in M_n(F)\mid e=e'\}$, and in the second case, y varies over $\{e\in M_{n-1}(F)\mid e=e'\}$. These integrals converge absolutely for $\mathrm{Re}(s)\gg 0$ and have a holomorphic continuation to the whole plane, which defines the Whittaker model for $\rho_{\tau,s}$ with respect to V_n and the character

(1.2)

$$\begin{pmatrix} z & x\\ 0 & z^*\end{pmatrix}\mapsto\begin{cases}\psi(z_{12}+z_{23}+\cdots+z_{n-1,n}-x_{n-1,1}), & n \text{ even},\\ \psi(z_{12}+z_{23}+\cdots+z_{n-2,n-1}-z_{n-1,n}+x_{n-1,1}), & n \text{ odd}.\end{cases}$$

Let

$$\widetilde{\xi}_{\tau^*,1-s}=M(w_n,\xi_{\tau,s}).$$

This is a section in the representation induced from $\tau^*\otimes|\det.|^{1/2-s}$ to H_n. The induction is from the parabolic subgroup Q_n if n is even, and from the parabolic subgroup $w_n\overline{Q}_n w_n^{-1}$ if n is odd. $\tau^*(m)=\tau(m^*)$. Denote this representation by $\widetilde{\rho}_{\tau^*,1-s}$. As for $\rho_{\tau,s}$, the following integrals define the Whittaker model of $\widetilde{\rho}_{\tau^*,1-s}$ with respect to the character (1.2),

$$W_{\widetilde{\xi}_{\tau^*,1-s}}(h)=\begin{cases}\int \widetilde{\xi}_{\tau^*,1-s}(w_n u(x)h,b_n^*)\psi(x_{n-1,1})dx, & n \text{ even},\\ \int \widetilde{\xi}_{\tau^*,1-s}(w_n u(x)h,b_{0,n}^*)\psi^{-1}(x_{n-1,1})dx, & n \text{ odd}.\end{cases}$$

The Shahidi local coefficient $\gamma(\tau,\Lambda^2,2s-1,\psi)$ is defined through the functional equation

$$\gamma(\tau,\Lambda^2,2s-1,\psi)W_{\widetilde{\xi}_{\tau^*,1-s}}(I)=W_{\xi_{\tau,s}}(I) \tag{1.3}$$

and we define $\gamma(\pi\times\tau,s,\psi)$ by

$$\Gamma(\pi\times\tau,s,\psi)=\frac{\gamma(\pi\times\tau,s,\psi)}{\gamma(\tau,\Lambda^2,2s-1,\psi)}. \tag{1.4}$$

2. Two realizations of $\rho_{\tau,s}$ for $\tau = \mathrm{Ind}_{P'_{1,n-1}}^{\mathrm{GL}_n(F)} \mu\otimes\tau'$

Let μ be a quasicharacter of F^* and τ' an admissible, finitely generated representation of $\mathrm{GL}_{n-1}(F)$, such that τ' admits a unique Whittaker model (thus $\gamma(\pi\times\tau',s,\psi)$ and $\gamma(\tau',\Lambda^2,s,\psi)$ are defined). We think of the elements of V_τ as smooth function $f(g;b)$ on $\mathrm{GL}_n(F)\times\mathrm{GL}_{n-1}(F)$, such that

$$f\Big(\begin{pmatrix}1 & v\\ & I_{n-1}\end{pmatrix}g;b\Big)=f(g;b),$$

$$(2.1)\qquad f\Big(\begin{pmatrix}a & 0\\ 0 & c\end{pmatrix}g;b\Big)=\frac{|a|^{(n-1)/2}}{|\det c|^{1/2}}\ \mu(a)f(g;bc);\quad a\in F^*,\ c\in\mathrm{GL}_{n-1}(F).$$

The function $m\mapsto f(g;m)$ lies in $W(\tau',\psi^{-1})$. In a similar way, we consider the elements of $V_{\rho_{\tau,s}}$ as smooth functions on $H_n\times\mathrm{GL}_n(F)\times\mathrm{GL}_{n-1}(F)$, $F(h,r,m)$, which satisfy

$$F(uh,r,b)=F(h,r,b),\quad u\in U_n,$$

$$(2.2)\qquad F(m(a)h,r,b)=F(h,ra,b),\quad a\in\mathrm{GL}_n(F),$$

$$F(h,\begin{pmatrix}a & v\\ 0 & c\end{pmatrix}r,b)=\mu(a)|a|^{s+n-3/2}|\det c|^{s+(n-3)/2}F(h,r,bc).$$

The function $b\mapsto F(h,r,b)$ lies in $W(\tau',\psi^{-1})$. We have the isomorphism

$$(2.3)\qquad \rho_{\tau,s}\simeq\mathrm{Ind}_{R_1}^{H_n}(\mu_s\otimes\rho_{\tau',s})$$

where

$$\mu_s(t)=\mu(t)|t|^{s-1/2}$$

($\rho_{\tau',s}$ is defined on H_{n-1} similar to $\rho_{\tau,s}$ on H_n.) We realize the elements of the r.h.s. of (2.3) as smooth functions $\phi(h,h',b)$ on $H_n\times H_{n-1}\times\mathrm{GL}_{n-1}(F)$, such that

$$\phi(yh,h',b)=\phi(h,h',b),\quad y\in U(R_1),$$

$$\phi\Big(\begin{pmatrix}x & & \\ & h'_0 & \\ & & x^{-1}\end{pmatrix}h,h',b\Big)=\mu(x)|x|^{s+n-3/2}\phi(h,h'h'_0,b),$$

$$x\in F^*,\ h'_0\in H_{n-1},$$

$$(2.4)\qquad \phi(h,uh',b)=\phi(h,h',b),\quad u\in U(Q_{n-1})\equiv U_{n-1},$$

$$\phi(h,m(a)h',b)=|\det a|^{s+(n-3)/2}\phi(h,h',ba),\quad a\in\mathrm{GL}_{n-1}(F).$$

The function $b\mapsto\phi(h,h',b)$ lies in $W(\tau',\psi^{-1})$.

The isomorphism (2.3) in terms of F and ϕ, which satisfy (2.2) and (2.4) respectively, is given by

$$\phi \mapsto F_\phi, \quad \text{where}$$

$$F_\phi(h,r,b) = \phi(m(r)h, I_{2n-2}, b). \tag{2.5}$$

Now let us compose F_ϕ with the Whittaker functional on τ,

$$\varphi_\phi(h,r) = \int_{F_{n-1}} F_\phi(h, w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix} r, I_{n-1})\psi(z_{n-1})dz. \tag{2.6}$$

For $k < n$, we denote

$$w_{k,n} = \begin{pmatrix} & I_k \\ I_{n-k} & \end{pmatrix}.$$

The integral (2.6) might not converge. To get convergence, we replace $\mu(x)$ by $\mu(x)|x|^\zeta$ and $\tau'(g)$ by $\tau'(g)|\det g|^{-\zeta}$ for $\mathrm{Re}(\zeta)$ large enough. Indeed, we have the following lemma whose proof is that of the analogous result for the similar intertwining integral.

LEMMA 2.1: *There is a positive number ζ_0, which depends on τ' and μ only, such that the integral (2.6) converges absolutely for $\mathrm{Re}(\zeta) \geq \zeta_0$, all ϕ and all s.*

To lighten our notation we do not denote $\phi_{\tau',\mu,s,\zeta}$ but rather just ϕ. Finally, define

$$f_\phi(h) = \varphi_\phi(h, I_n). \tag{2.7}$$

Note that φ_ϕ is a $\xi_{\tau,s}$ and f_ϕ is an $f_{\xi_{\tau,s}}$ in the notation of Section 1. Now we are ready to substitute φ_ϕ for $\xi_{\tau,s}$ in $A(W,\xi_{\tau,s})$.

3. Proof of Theorem 2 in case $r = n - \ell - 1 \geq 1$

We prove directly the identity

$$\frac{\gamma(\pi\times\mu, s+\zeta,\psi)\gamma(\pi\times\tau', s-\zeta,\psi)}{\gamma(\tau'\times\mu, s-\frac{1}{2},\psi)\gamma(\tau',\Lambda^2, 2(s-\zeta)-1,\psi)} A(W,\varphi_\phi) = \widetilde{A}(W,\varphi_\phi) \tag{3.1}$$

in case $r \geq 1$.

The factor $\gamma(\tau'\times\mu, s-\frac{1}{2})$ is the gamma factor for $GL_{n-1}\times GL_1$, which also equals the corresponding local coefficient of Shahidi.

a. Direct Substitution of φ_ϕ in $A(W,\varphi_\phi)$. This results in

$$A(W,\varphi_\phi)=\int_{N_\ell\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}}\int_{F_{n-1}}\phi\Big(m(w_{1,n}\begin{pmatrix} I_{n-1} & z\\ & 1\end{pmatrix})$$

$$\overline{x}\beta_{\ell,n}i_{\ell,n}(g),I_{2n-2},I_{n-1})\psi(z_{n-1})\psi_a(\overline{x})dzd\overline{x}dg. \tag{3.2}$$

Lemma 3.1: *The integral (3.2) converges absolutely as a triple integral in a domain of the form*

$$A\le \mathrm{Re}(\zeta)\le \mathrm{Re}(s)+B, \tag{3.3}$$

where the constants A,B depend only on π,τ' and μ.

The lemma is proved in Section 6.a.

b. Lemma 3.2: *We have, in the domain (3.3),*

$$A(W,\varphi_\phi)=\int_{N_\ell\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n-1)}}\int_{F_{2n-2}}\phi\Big(\begin{pmatrix}1 & & \\ v & I_{2n-2} & \\ * & v' & 1\end{pmatrix}$$

$$m(w_{1,n})\beta_{\ell,n},\overline{x}i_{\ell,n-1}(g),I_{n-1}\Big)\psi(v_{n-1})\psi_a(\overline{x})d(v,\overline{x},g). \tag{3.4}$$

Here ψ_a is adapted to $\overline{X}^{(\ell,n-1)}$.

Proof: By a simple change of variables, we may replace $m\begin{pmatrix} I_{n-1} & z\\ & 1\end{pmatrix}\overline{x}$ by $\overline{x}m\begin{pmatrix} I_{n-1} & z\\ & 1\end{pmatrix}$ in (3.2) (in the domain (3.3)). Write $\overline{x}=\overline{x}'\overline{x}''$, where

$$\overline{x}'=\overline{u}\begin{pmatrix} \overbrace{0}^{\ell+1} & \overbrace{0}^{r-1} & \overbrace{0}^{1}\\ v_2 & u_2 & 0\\ 0 & v_2' & 0\end{pmatrix}\begin{matrix}\}\,1\\ \}\,r-1\\ \}\,\ell+1\end{matrix}$$

and

$$\overline{x}''=\overline{u}\begin{pmatrix} \overbrace{v_1}^{\ell+1} & \overbrace{u_1}^{r-1} & \overbrace{0}^{1}\\ 0 & 0 & u_1'\\ 0 & 0 & v_1'\end{pmatrix}\begin{matrix}\}\,1\\ \}\,r-1\\ \}\,\ell+1\end{matrix}.$$

We have

$$m(w_{1,n})\overline{x}'m(w_{1,n})^{-1} = \overline{u}\begin{pmatrix} 0 & v_2 & u_2 \\ 0 & 0 & v_2' \\ 0 & 0 & 0 \end{pmatrix},$$

$$m(w_{1,n})\overline{x}''m(w_{1,n})^{-1} = \overline{u}\begin{pmatrix} u_1' & 0 & 0 \\ v_1' & 0 & 0 \\ 0 & v_1 & u_1 \end{pmatrix},$$

$$w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1 \end{pmatrix} w_{1,n}^{-1} = \begin{pmatrix} 1 & \\ z & I_{n-1} \end{pmatrix},$$

$$m(w_{1,n})i_{\ell,n}(g)m(w_{1,n})^{-1} = \begin{pmatrix} 1 & & \\ & i_{\ell,n-1}(g) & \\ & & 1 \end{pmatrix}. \tag{3.5}$$

Using (3.5), (2.4) (and the fact that $\beta_{\ell,n}$ commutes with $i_{\ell,n}(g)$), we get (3.4) from (3.2). Note that if $r > 1$ then the conjugation of $\begin{pmatrix} 1 & & \\ v & I_{m-2} & \\ * & v' & 1 \end{pmatrix}$ by $\begin{pmatrix} 1 & & \\ & i_{\ell,n-1}(g) & \\ & & 1 \end{pmatrix}$ does not affect $\psi(v_{n-1})$. If $r = 1$, we have $\overline{x} = \overline{x}''$, $\overline{x}' = I_{4n}$ and $\psi(z_{n-1})\psi_a(\overline{x})$ becomes $\psi(v_{n-1} - v_n)$. This character is preserved by $\begin{pmatrix} 1 & & \\ & i_{\ell,n-1}(g) & \\ & & 1 \end{pmatrix}$ $(n - \ell - 1 = 1)$. ∎

The integral (3.4) converges absolutely in the domain (3.3). If we consider its $d\overline{x}dg$ integration (on $N_\ell \backslash G_\ell \times \overline{X}^{(\ell,n-1)}$) first, we recognize a local integral for $G_\ell \times \mathrm{GL}_{n-1}(F)$ and $\pi \times \tau'$. (There is a missing translation by $\beta_{\ell,n-1}$.)

c. Applying the Functional Equation for $\pi \times \tau'$. A formal application of this functional equation to the $d(\overline{x}, g)$ integration in (3.4) gives

$$\frac{\gamma(\pi \times \tau', s-\zeta, \psi)}{\gamma(\tau', \Lambda^2, 2(s-\zeta)-1, \psi)} A(W, \varphi_\phi) = \int_{F_{2n-2}} \int_{N_\ell \backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n-1)}}$$

$$\phi^{\sim}\left(\begin{pmatrix} 1 & & \\ v & I_{2n-2} & \\ * & v' & 1 \end{pmatrix} m(w_{1,n})\beta_{\ell,n}, \overline{x}\alpha_{\ell,n-1}i_{\ell,n-1}(g), I_{n-1}\right) \tag{3.6}$$

$$\cdot\, \psi(v_{n-1})\psi_a^{(-1)^{n-1}}(\overline{x})d\overline{x}dgdv.$$

Let us first explain the notation in (3.6). Put, for $h \in H_n$,

$$\phi_h(h', b) = \phi(h, h', b), \quad h' \in H_{n-1},\ b \in \mathrm{GL}_{n-1}(F),$$

ϕ_h lies in $V_{\rho_{\tau',s-\zeta}}$. Then

$$\phi^{\sim}(h,h',b)=\begin{cases} M(w_{n-1},\phi_h)(h',b^*_{\ell,n-1}b^*), & n \text{ odd},\\ M(w_{n-1},\phi_h)(h^{w_{n-1}},b^*_{\ell,n-1}b^*), & n \text{ even};\end{cases}$$

$$\alpha_{\ell,n-1}=\begin{cases} I_{2n-2}, & n \text{ odd},\\ \eta_{\ell,n-1}m(\varepsilon_{\ell,n-1})\beta^{w_{n-1}}_{\ell,n-1}, & n \text{ even}.\end{cases}$$

See Section 1 for the notation. Now let us explain how to interpret (3.6). In Section 6.b we prove

LEMMA 3.3: *The integral (3.6) converges absolutely (as a multiple integral) in a domain of the form*

$$\widetilde{A}\leq \mathrm{Re}(s)\leq \mathrm{Re}(\zeta)+\widetilde{B}, \tag{3.7}$$

where $\widetilde{A},\widetilde{B}$ are constants which depend only on π,τ' and μ.

The domains (3.3) and (3.7) might be disjoint. We follow the same reasoning as in [S1, Sect. 11]. The integral (3.6), which we denote by $B(W,\varphi_\phi)$, has a meromorphic continuation to the whole plane and is a rational function in q^{-s} (fix ζ). (This follows from [S1, 8.4] since the integral (3.6) clearly satisfies the equivariance property (1.3.2) of [S1].) By [S1, 8.3], $B(W,\varphi_\phi)$ is proportional to $A(W,\varphi_\phi)$ by a meromorphic function of s (and actually of ζ as well.) More precisely, we have

$$\frac{c(\pi,\tau',s-\zeta,\psi)}{\gamma(\tau',\Lambda^2,2(s-\zeta)-1,\psi)}A(W,\varphi_\phi)=B(W,\varphi_\phi)$$

where $c(\pi,\tau',x,\psi)$ is rational in q^{-x}. To find c, it is enough to compute $A(W,\varphi_\phi)$ and $B(W,\varphi_\phi)$ for a special substitution of W and ϕ. This is shown in Section 6.c and we, of course, get

LEMMA 3.4:

$$c(\pi,\tau',s-\zeta,\psi)=\gamma(\pi\times\tau',s-\zeta,\psi).$$

d. UNFOLDING $B(W,\varphi_\phi)$ BACK. We unfold $B(W,\varphi_\phi)$ "back" from (3.6) to an integral similar to (3.2). This we do in the domain (3.7), where the rational function $B(W,\varphi_\phi)$ is represented by the convergent integral (3.6).

LEMMA 3.5: *We have, in the domain (3.7),*

$$\begin{aligned} B(W,\varphi_\phi)=&\int_{N_\ell\backslash G_\ell}W(g)\int_{\overline{X}^{(\ell,n)}}\\ &\int_{F_{n-1}}\widetilde{\phi}\left(m(w_{1,n}\begin{pmatrix} I_{n-1} & z\\ & 1\end{pmatrix})\overline{x}\delta_{\ell,n}i_{\ell,n}(g),I_{2n-2},I_{n-1}\right)\\ &\cdot\psi(z_{n-1})\psi_a^{(-1)^{n-1}}(\overline{x})d(z,\overline{x},g).\end{aligned}\tag{3.8}$$

Here

$$\widetilde{\phi}(h,h',b)=\begin{cases}\phi^{\sim}(h,h',b), & n \text{ odd},\\ \phi^{\sim}(h^{\omega_n},h',b), & n \text{ even},\end{cases}$$

$h\in H_n, h'\in H_{n-1}, b\in \mathrm{GL}_{n-1}(F)$ *and*

$$\delta_{\ell,n}=\begin{cases}\beta_{\ell,n}, & n \text{ odd},\\ m(w_{1,n})^{-1}\begin{pmatrix}1&&\\&\alpha_{\ell,n-1}&\\&&1\end{pmatrix}(m(w_{1,n})\beta_{\ell,n})^{\omega_n}, & n \text{ even}.\end{cases}$$

Proof: If n is odd, then $\alpha_{\ell,n-1}=I_{2n-2}$ and (3.8) is obtained from (3.6) by reversing the steps which led from (3.2) to (3.4). Assume that n is even. We have

$$\alpha_{\ell,n-1}^{\omega_{n-1}}=\begin{cases}\left(\begin{array}{c|c|c|c|c|c|c|c} I_\ell &&&&&&& \\ \hline & 0 &&&&& 1 & \\ \hline && I_{r-2} &&&&& \\ \hline &&& 0 & 1 &&& \\ \hline &&& 1 & 0 &&& \\ \hline &&&&& I_{r-2} && \\ \hline & 1 &&&&& 0 & \\ \hline &&&&&&& I_\ell \end{array}\right), & r\geq 3,\\ \left(\begin{array}{c|cccc|c} I_\ell &&&&& \\ \hline &&&& 1 & \\ &&& 1 && \\ && 1 &&& \\ & 1 &&&& \\ \hline &&&&& I_\ell \end{array}\right), & r=2,\\ I_{2n-2}, & r=1a.\end{cases}$$

Using this and some of the steps which led from (3.2) to (3.4), we see that

$$B(W,f_\phi)=\int_{F_{2n-2}}\int_{N_\ell\backslash G_\ell}W(g)\int_{\overline{X}^{(\ell,n-1)}}$$
$$\phi^{\sim}(\begin{pmatrix}1&&\\&\overline{x}^{\omega_{n-1}}&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\v&I_{2n-2}&\\ *&v'&1\end{pmatrix}\begin{pmatrix}1&&\\&\alpha_{\ell,n-1}^{\omega_{n-1}}&\\&&1\end{pmatrix}$$
$$\cdot\, m(w_{1,n})\beta_{\ell,n}i_{\ell,n}(g), I_{2n-2}, I_{n-1})\cdot\psi(v_n)\psi_a^{-1}(\overline{x})d\overline{x}dgdv=$$

(note that in case $r>1$, conjugation of $\begin{pmatrix}1&&\\v&I_{2n-2}&\\ *&v'&1\end{pmatrix}$ by $\begin{pmatrix}1&&\\&\alpha_{\ell,n-1}^{\omega_{n-1}}&\\&&1\end{pmatrix}$

takes v_{n-1} to v_n)

$$= \int_{F_{2n-2}} \int_{N_\ell \backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n-1)}} \tilde{\phi}\Big(\begin{pmatrix} 1 & & \\ & \overline{x} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ v & I_{2n-2} & \\ * & v' & 1 \end{pmatrix}$$

$$\cdot m(w_{1,n})\delta_{\ell,n} i_{\ell,n}(g), I_{2n-2}, I_{n-1}\Big)\psi(v_{n-1})\psi_a^{-1}(\overline{x}) dv d\overline{x} dg.$$

Note that $\tilde{\phi}$ satisfies (2.4) with the following changes. In the second property (of (2.4)) replace s by $s+\zeta$. In the fourth property replace s by $1-(s-\zeta)$. In the fifth property replace $W(\tau', \psi^{-1})$ by $W(\tau'^*, \psi^{-1})$ in case n is odd and by $W(\tau'^*, \psi^*)$ in case n is even, where

$$\psi^* \begin{pmatrix} 1 & z_1 & & & * \\ & 1 & z_2 & & \\ & & \ddots & & \\ & & & 1 & z_{n-2} \\ & & & & 1 \end{pmatrix} = \psi^{-1}(z_1 + z_2 + \cdots - z_{\ell+1} + \cdots + z_{n-2}).$$

If $r=1$, $\psi^* = \psi$. In detail, we have

$$\tilde{\phi}(yh, h', b) = \phi(h, h', b), \quad y \in U(R_1),$$

$$\tilde{\phi}\left(\begin{pmatrix} x & & \\ & h'_0 & \\ & & x^{-1} \end{pmatrix} h, h', b \right) = \mu(x)|x|^{s+\xi+n-\frac{3}{2}} \tilde{\phi}(h, h'h'_0, b),$$

$$x \in F^*, h'_0 \in H_{n-1},$$

$$\tilde{\phi}(h, nh', b) = \phi(h, h', b), \quad u \in U(Q_{n-1}) \equiv U_{n-1},$$

$$\tilde{\phi}(h, m(a)h', b) = |\det a|^{1-(s-\xi)+(n-3)/2} \tilde{\phi}(h, h', ba), \quad a \in \mathrm{GL}_{n-1}(F).$$

The function $b \mapsto \tilde{\phi}(h, h', b)$ lies in $W(\tau'^*, \psi^{-1})$ in case n is odd and in $W(\tau'^*, \psi^*)$ in case n is even.

Thus we can use (in the reverse direction) the set of the steps which led from (3.2) to (3.4) to conclude (3.8). ∎

e. Application of Shahidi's Functional Equation for $\tau' \times \mu$. By the definition of $F_{\tilde{\phi}}$ (see (2.5)), we rewrite (3.8) in the domain (3.3) as

$$B(W, \varphi_\phi) = \int_{N_\ell \backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} \int_{F_{n-1}} F_{\tilde{\phi}}(\overline{x}\delta_{\ell,n} i_{\ell,n}(g), w_{1,n} \begin{pmatrix} I_{n-1} & z \\ & 1 \end{pmatrix}, I_{n-1})$$

$$(3.9) \qquad \psi(z_{n-1})\psi_a^{(-1)^{n-1}}(\overline{x}) dz d\overline{x} dg.$$

At this point we use Shahidi's functional equation and local coefficient for $GL_{n-1}\times GL_1$. It is defined as follows. Let σ be an irreducible generic representation of $GL_{n-1}(F)$ and η a quasi-character of F^*. Put $\eta_s(x)=\eta(x)|x|^{s-\frac{1}{2}}$ and $\sigma_s(r)=\sigma(r)|\det r|^{s-\frac{1}{2}}$. Consider the representation

$$\pi_{\eta,\sigma,s_1,s_2}=\mathrm{Ind}_{P'_{1,n-1}}^{GL_n(F)}\eta_{s_1+\frac{n-1}{2}}\otimes\sigma_{s_2+\frac{n}{2}-1}.$$

Assume that σ is realized in its standard Whittaker model with respect to ψ^{-1}. We think of an element e of $\pi_{\eta,\sigma,s_1,s_2}$ as a function on $GL_n(F)\times GL_{n-1}(F)$, $e(m,r)$ such that $r\mapsto e(m,r)$ is in $W(\sigma,\psi^{-1})$. Consider the intertwining operator given by

$$\widetilde{e}(m,r)=\int_{F_{n-1}}e(w_{1,n}\begin{pmatrix}I_{n-1}&z\\&1\end{pmatrix}m,r)dz$$

and the following Whittaker models:

$$W_e(m)=\int_{F_{n-1}}e(w_{1,n}\begin{pmatrix}I_{n-1}&z\\&1\end{pmatrix}m,I_{n-1}))\psi(z_{n-1})dz,$$

$$W_{\widetilde{e}}(m)=\int_{F^{n-1}}\widetilde{e}\left(w_{1,n}^{-1}\begin{pmatrix}1&t\\&I_{n-1}\end{pmatrix}m,I_{n-1}\right)\psi(t_1)dt.$$

These are models with respect to the character ψ^{-1} (of Z_n). We consider the local coefficient $c_\psi(\eta_{s_1+(n-1)/2}\otimes\sigma_{s_2+n/2-1})$ defined by

$$c_\psi(\eta_{s_1+(n-1)/2}\otimes\sigma_{s_2+n/2-1})W_{\widetilde{e}}(m)=W_e(m). \tag{3.10}$$

We have, by the multiplicativity of local coefficients,

(3.11)

$$\gamma(\tau,\Lambda^2,2s-1,\psi)=c_\psi(\mu_{s+\zeta+(n-1)/2}\otimes(\tau')^*_{-(s-\zeta)+n/2})\gamma(\tau',\Lambda^2,2(s-\zeta)-1,\psi).$$

By (3.10), we get from (3.9)

$$B(W,\varphi_\phi)=c_\psi(\mu_{s+\zeta+(n-1)/2}\otimes(\tau')^*_{-(s-\zeta)+n/2})\int_{N_\ell\backslash G_\ell}W(g)\int_{\overline{X}^{(\ell,n)}}\int_{F^{n-1}}$$

$$(\widetilde{F_{\tilde\phi}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(g),w_{1,n}^{-1}\begin{pmatrix}1&t\\&I_{n-1}\end{pmatrix}\begin{pmatrix}I_{\ell+1}&&\\&(-1)^{n-1}&\\&&I_{r-1}\end{pmatrix};I_{n-1}\right).$$

$$\psi(t_1)\psi_a^{(-1)^{n-1}}(\overline{x})dtd\overline{x}dg. \tag{3.12}$$

Here

$$(\widetilde{F_{\tilde\phi}})(h,m,r)=\int_{F_{n-1}}F_{\tilde\phi}\left(h,w_{1,n}\begin{pmatrix}I_{n-1}&z\\&1\end{pmatrix}m,r\right)dz, \tag{3.13}$$

the composition of $F_{\tilde{\phi}}$ with the intertwining operator on τ. Of course we take (3.13) in the sense of analytic continuation. In Section 6.d we prove

LEMMA 3.6: *The integral (3.12) converges absolutely (as a multiple integral) in a domain of the form*

$$-\operatorname{Re}(\zeta)+L\leq \operatorname{Re}(s)\leq \operatorname{Re}(\zeta)+L',$$
$$\operatorname{Re}(s)\leq M \tag{3.14}$$

for some constants L, L', M which depend only on π, τ' and μ.

As in Lemma 3.4, we conclude that there is a meromorphic function $d(\pi,\tau,s,\zeta,\psi)$ such that equality (3.12) holds (as meromorphic functions) with the local coefficient replaced by $d(\pi,\tau,s,\zeta,\psi')$, and we prove in Section 6.e

LEMMA 3.7:

$$d(\pi,\tau,s,\zeta,\psi)=c_\psi(\mu_{s+\zeta+(n-1)/2}\times(\tau')^*_{-(s-\zeta)+n/2}).$$

We proved that

$$\frac{\gamma(\pi\times\tau',s-\zeta,\psi)}{c_\psi(\mu_{s+\zeta+(n-1)/2}\times(\tau')^*_{-(s-\zeta)+n/2})\gamma(\tau',\Lambda^2,2(s-\zeta)-1,\psi)}A(W,\varphi_\phi)=C(W,\phi),$$

i.e., by (3.11)

$$\frac{\gamma(\pi\times\tau',s-\zeta,\psi)}{\gamma(\tau,\Lambda^2,2(s-\zeta)-1,\psi)}A(W,\varphi_\phi)=C(W,\phi).$$

Here $C(W,\phi)$ is the integral on the r.h.s. of (3.12).

f. LEMMA 3.8: *In the domain (3.14), we have*

$$C(W,\phi)=\int_{E_\ell N_{\ell-1}\backslash G_\ell}W(g)\int_{\overline{X}^{(\ell,n)}}\int_Z(\widetilde{F_{\tilde{\phi}}})\left(\bar{x}\delta_{\ell,n}i_{\ell,n}(g),w_{1,n}^{-1}z\begin{pmatrix}I_{\ell+1}&&\\&(-1)^{n-1}&\\&&I_{r-1}\end{pmatrix},I_{n-1}\right)\psi_a^{(-1)^{n-1}}(\bar{x})\,dz\,d\bar{x}\,dg. \tag{3.15}$$

Here E_ℓ is the subgroup of matrices of the form $\begin{pmatrix}1&0&0&*&0\\&I_{\ell-1}&0&0&*\\&&1&0&0\\&&&I_{\ell-1}&0\\&&&&1\end{pmatrix}$ in G_ℓ; Z is the subgroup of matrices of the form $\begin{pmatrix}1&0&*\\&I_\ell&0\\&&I_r\end{pmatrix}$ in $\mathrm{GL}_n(F)$.

Proof: Write

$$\begin{pmatrix}1 & t\\ & I_{n-1}\end{pmatrix}=\begin{pmatrix}1 & t' & 0\\ & I_{\ell-1} & 0\\ & & I_{n-\ell}\end{pmatrix}\begin{pmatrix}1 & 0 & t''\\ & I_{\ell-1} & 0\\ & & I_{n-\ell}\end{pmatrix}\equiv z'\cdot z''.$$

Let Z' and Z'' be the corresponding subgroups. Let

$$E'_\ell=\left\{e(z)=\begin{pmatrix}1 & z & *\\ & I_{2\ell-1} & z'\\ & & 1\end{pmatrix}\in G_\ell \mid z\in F^{2\ell-1}\right\},$$
$$E''_\ell=\{e(z)\mid z_1=\cdots=z_{\ell-1}=0\}.$$

We have

$$N_\ell=E'_\ell\cdot N_{\ell-1},\quad E'_\ell=\hat{Z}'E''_\ell.$$

(Here $N_{\ell-1}$ is already embedded as $\begin{pmatrix}1 & & \\ & N_{\ell-1} & \\ & & 1\end{pmatrix}$ inside G_ℓ; Z' is considered as a subgroup of $\mathrm{GL}_\ell(F)$ as well as a subgroup of $\mathrm{GL}_n(F)$ and $i_{\ell,n}(\hat{z}')=m(z')$.) We have

$$m(z')\overline{X}^{(\ell,n)}m(z')^{-1}=\overline{X}^{(\ell,n)},$$
$$\psi_a(m(z')\overline{x}m(z')^{-1})=\psi_a(\overline{x}),$$
$$d(m(z')\overline{x}m(z')^{-1})=d\overline{x},$$
$$m(z')\delta_{\ell,n}=\delta_{\ell,n}m(z'),$$

and so

$$C(W,\phi)=\int_{E''_\ell N_{\ell-1}\backslash G_\ell}W(g)\int_{\overline{X}^{(\ell,n)}}\int_{Z''}(\widetilde{F_{\tilde\phi}})\Bigg(\overline{x}\delta_{\ell,n}i_{\ell,n}(g),w_{1,n}^{-1}z''$$
$$\tag{3.16}\begin{pmatrix}I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1}\end{pmatrix},I_{n-1}\Bigg)\cdot\psi_a^{(-1)^{n-1}}(\overline{x})dz''d\overline{x}dg.$$

Write

$$z''=z_yz,$$

where

$$z_y=\begin{pmatrix}1 & 0 & y & 0\\ & I_{\ell-1} & 0 & 0\\ & & 1 & 0\\ & & & I_r\end{pmatrix},\quad z=\begin{pmatrix}1 & 0 & *\\ & I_\ell & 0\\ & & I_r\end{pmatrix}.$$

We have for $\overline{x} = \begin{pmatrix} I_\ell & & & & & \\ & 1 & & & & \\ & & I_r & & & \\ v & u & \nu & I_r & & \\ 0 & 0 & u' & & 1 & \\ 0 & 0 & v' & & & I_\ell \end{pmatrix}$,

$$\psi_a\Big(m(z_y)\overline{x}m(z_y^{-1})\Big) = \psi_a(\overline{x})\psi^{-1}(yv_{r,1}).$$

Thus from (3.16), we get

$$C(W,\phi) = \int_{E_\ell'' N_{\ell-1}\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} \int_F \int_Z (\widetilde{F_{\tilde{\phi}}})\Big(\overline{x}m(z_y)\delta_{\ell,n}i_{\ell,n}(g), w_{1,n}^{-1}z \begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1} \end{pmatrix}, I_{n-1})\Big)$$
$$\cdot \psi_a^{(-1)^{n-1}}(\overline{x})\psi^{(-1)^{n-1}}(yv_{r,1})dzdyd\overline{x}dg. \tag{3.17}$$

Note that v remains the same for $\overline{x}$ and for $m(z_y)\overline{x}m(z_y)^{-1}$. We have

$$\delta_{\ell,n}^{-1}m(z_y)\delta_{\ell,n} = \begin{cases} m(z_y), & \ell \text{ even} \\ u_y, & \ell \text{ odd} \end{cases}$$

where

$$u_y = u \begin{pmatrix} \overbrace{0}^{r} & \overbrace{y}^{1} & \overbrace{0}^{\ell-1} & \overbrace{-\frac{1}{2}y^2}^{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We also have

$$\delta_{\ell,n}^{-1}u_y\delta_{\ell,n} = \begin{cases} u_y, & \ell \text{ even,} \\ m(z_y), & \ell \text{ odd.} \end{cases}$$

Thus, for ℓ even, we have in (3.17)

$$(\widetilde{F_{\tilde{\phi}}})\Big(\overline{x}m(z_y)\delta_{\ell,n}i_{\ell,n}(g),\dots\Big) = (\widetilde{F_{\tilde{\phi}}})\Big(u_{-y}\overline{x}u_{-y}^{-1}\delta_{\ell,n}u_{-y}m(z_y)i_{\ell,n}(g),\dots\Big),$$

and for ℓ odd,

$$(\widetilde{F_{\tilde{\phi}}})\Big(\overline{x}m(z_y)\delta_{\ell,n}i_{\ell,n}(g),\dots\Big) = (\widetilde{F_{\tilde{\phi}}})\Big(u_{-y}\overline{x}u_{-y}^{-1}\delta_{\ell,n}m(z_{-y})u_yi_{\ell,n}(g),\dots\Big).$$

Note that

$$u_{-y}m(z_y)=m(z_y)u_{-y}=i_{\ell,n}\left(\begin{pmatrix}1&0&y&0&\frac{1}{2}y^2\\ &I_{\ell-1}&0&0&0\\ &&1&0&-y\\ &&&I_{\ell-1}&0\\ &&&&1\end{pmatrix}\right).$$

We have (with previous notation)

$$u_{-y}\overline{x}u_{-y}^{-1}=m\begin{pmatrix}I_\ell&0&*\\ &1&-y(v_{r,1}\dots v_{1,1})\\ &&I_r\end{pmatrix}\overline{u}\begin{pmatrix}v&u&\nu+\cdots\\ 0&0&u'\\ 0&0&v'\end{pmatrix}.$$

Thus

$$C(W\phi)=\int_{E_\ell N_{\ell-1}\backslash G_\ell}W(g)\int_{\overline{X}^{(\ell,n)}}\int_Z(\widetilde{F_{\tilde\phi}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(g),w_{1,n}^{-1}z\begin{pmatrix}I_{\ell+1}&&\\ &(-1)^{n-1}&\\ &&I_{r-1}\end{pmatrix},I_{n-1}\right)$$

$$\cdot\psi_a^{(-1)^{n-1}}(\overline{x})dzd\overline{x}dg. \tag{3.18}$$ ∎

g. Factoring Integration Through $\begin{pmatrix}*&\\ *&I_{\ell-1}\end{pmatrix}^\wedge$.

Lemma 3.9: *In the domain (3.14), we have*

$$C(W,\phi)=\int_{\hat C_\ell E_\ell N_{\ell-1}\backslash G_\ell}\left(\int_{F^*}\int_{\overline{X}_{(1,\ell)}}W\left(\overline{y}j_{1,\ell}\begin{pmatrix}t&\\ &t^{-1}\end{pmatrix}g\right)\mu(t)|t|^{s+\zeta-1/2}d\overline{y}d^*t\right)\cdot\int_{\overline{X}^{(\ell,n)}}\int_Z(\widetilde{F_{\tilde\phi}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(g),w_{1,n}^{-1}z\begin{pmatrix}I_{\ell+1}&&\\ &(-1)^{n-1}&\\ &&I_{r-1}\end{pmatrix},I_{n-1}\right)\psi_a^{(-1)^{n-1}}(\overline{x})dzd\overline{x}dg. \tag{3.19}$$

Here C_ℓ is the subgroup $\begin{pmatrix}*&\\ *&I_{\ell-1}\end{pmatrix}$ of $GL_\ell(F)$ and the dg integration of (3.19) should be understood in the sense of Iwasawa decomposition. (Recall that $\hat C_\ell$ is the image of C_ℓ in G_ℓ as explained in the notation.)

Proof: Factor the dg integration (in the above sense) in (3.18) through $\hat C_\ell$.

Write $c_{t,y} = \begin{pmatrix} t & 0 \\ y & I_{\ell-1} \end{pmatrix}$. We have

$$(\widetilde{F_{\tilde{\phi}}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(\hat{c}_{t,y}g), w_{1,n}^{-1}z\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1} \end{pmatrix}, I_{n-1}\right) =$$

$$\mu(t)|t|^{s+\zeta-1/2}(\widetilde{F_{\tilde{\phi}}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(g), w_{1,n}^{-1}z\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1} \end{pmatrix}, I_{n-1}\right)$$

and the lemma follows. ∎

h. Application of the Functional Equation for $\pi \times \mu$. The $d\overline{y}d^*t$ integration in (3.19) is a local integral for $\pi \times \mu$ (with s replaced by $s+\zeta$). A formal application of the functional equation in this case yields (going back to Section e)

$$\frac{\gamma(\pi \times \mu, s+\zeta, \psi)\gamma(\pi \times \tau', s-\zeta, \psi)}{\gamma(\tau, \Lambda^2, 2s-1, \psi)} A(W, \varphi_\phi) = D(W, \phi) \tag{3.20}$$

where

$$D(W,\phi) = \int_{\hat{C}_\ell E_\ell N_{\ell-1}\backslash G_\ell} \left(\int_{F^*} \int_{\overline{X}_{(1,\ell)}} W(\hat{c}_{1,\ell}\overline{y}j_{1,\ell}\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} a_{1,\ell}g)\mu^{-1}(t) \right.$$
$$\left. |t|^{1/2-(s+\zeta)}d\overline{y}d^*t\right) \cdot \int_{\overline{X}^{(\ell,n)}} \int_Z (\widetilde{F_{\tilde{\phi}}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(g), w_{1,n}^{-1}z\right.$$
$$\left.\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1} \end{pmatrix}, I_{n-1}\right) \cdot \psi_a^{(-1)^{n-1}}(\overline{x})dzd\overline{x}dy.$$

Recall that $a_{1,\ell} = \begin{pmatrix} & & 1 \\ & I_{2\ell-1} & \\ 1 & & \end{pmatrix}\begin{pmatrix} I_\ell & & \\ & -1 & \\ & & I_\ell \end{pmatrix}$ and $c_{1,\ell} = \begin{pmatrix} 1 & \\ & -I_{\ell-1} \end{pmatrix}$.
In Section 6.f, we prove

Lemma 3.10: *The integral (3.20) converges absolutely in a domain of the form*

$$\mathrm{Re}(s) + R \le \mathrm{Re}(\zeta) \le T \tag{3.21}$$

where the constants R, T depend only on π, τ' and μ.

i. Unfolding $D(W,\phi)$ Back. We unfold $D(W,\phi)$ "back" to an integral similar to (3.18).

LEMMA 3.11: *In the domain (3.21), we have*

$$D(W,\phi)=\int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}\times Z}(\widetilde{F_{\tilde\phi}})\Bigg(\overline{x}\delta_{\ell,n}i_{\ell,n}(e_\ell g),w_{1,n}^{-1}z$$

$$\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1}\end{pmatrix},I_{n-1}\Bigg)\cdot\psi_a^{(-1)^{n-1}}(\overline{x})dzd\overline{x}dg. \tag{3.22}$$

Here $e_\ell=a_{1,\ell}\hat{c}_{1,\ell}=\begin{pmatrix} & & 1\\ & -I_{2\ell-1} & \\ 1 & & \end{pmatrix}$.

Proof: As in Lemma 3.9, we have

$$D(W,\phi)=\int_{\overline{X}_{(1,\ell)}E_\ell N_{\ell-1}\backslash G_\ell}\int_{\overline{X}_{(1,\ell)}} W(\overline{y}e_\ell g)\int_{\overline{X}^{(\ell,n)}\times Z}(\widetilde{F_{\tilde\phi}})\Bigg(\overline{x}\delta_{\ell,n}i_{\ell,n}(g),w_{1,n}^{-1}z$$

$$\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{n-1}\end{pmatrix},I_{n-1}\Bigg)\cdot\psi_a^{(-1)^{n-1}}(\overline{x})d\overline{y}dzd\overline{x}dg.$$

We have, for $\overline{y}=m\begin{pmatrix}1 & \\ y & I_{\ell-1}\end{pmatrix}$,

$$e_\ell^{-1}\overline{y}e_\ell=\begin{pmatrix}1 & 0 & 0 & -y' & 0\\ & I_{\ell-1} & 0 & 0 & -y\\ & & 1 & 0 & 0\\ & & & I_{\ell-1} & 0\\ & & & & 1\end{pmatrix}\equiv\widetilde{y}. \tag{3.23}$$

Note that

$$i_{\ell,n}(\widetilde{y})=u\begin{pmatrix}0 & -y' & 0\\ 0 & 0 & -y\\ 0 & 0 & 0\end{pmatrix}\quad\text{and}\quad i_{\ell,n}(\widetilde{y})\delta_{\ell,n}=\delta_{\ell,n}i_{\ell,n}(\widetilde{y}).$$

Thus

$$(\widetilde{F_{\tilde\phi}})\Big(\overline{x}\delta_{\ell,n}i_{\ell,n}(g),\dots\Big)=(\widetilde{F_{\tilde\phi}})\Big(i_{\ell,n}(\widetilde{y})\overline{x}\delta_{\ell,n}i_{\ell,n}(g),\dots\Big). \tag{3.24}$$

We have for $\overline{x}=\overline{u}\begin{pmatrix}\overbrace{v_1}^{1} & \overbrace{v}^{\ell-1} & \overbrace{b}^{1} & \overbrace{c}^{r}\\ 0 & 0 & 0 & b'\\ 0 & 0 & 0 & v'\\ 0 & 0 & 0 & v_1'\end{pmatrix}\begin{matrix}\}\,r\\ \}\,1\\ \}\,\ell-1\\ \}\,1\end{matrix}$

$i_{\ell,n}(\widetilde{y})\overline{x}i_{\ell,n}(\widetilde{y})^{-1}=$

$$(3.25)\qquad m\begin{pmatrix}1&0&0&-y'v'\\ &I_{\ell-1}&0&-yv_1'\\ &&1&0\\ &&&I_r\end{pmatrix}\overline{u}\begin{pmatrix}v_1&v&b&c-v_1y'v'-v'yv_1\\ 0&0&0&b'\\ 0&0&0&v'\\ 0&0&0&v_1'\end{pmatrix}.$$

Using (3.23)–(3.25), we get

$$D(W,\phi)=\int_{\overline{X}_{(1,\ell)}E_\ell N_{\ell-1}\backslash G_\ell}\int_{F_{\ell-1}}W(e_\ell\widetilde{y}g)\int_{\overline{X}^{(\ell,n)}\times Z}$$

$$(\widetilde{F_{\tilde\phi}})\left(i_{\ell,n}(\widetilde{y})\overline{x}i_{\ell,n}(\widetilde{y})^{-1}\delta_{\ell,n}i_{\ell,n}(\widetilde{y}g),\dots\right)=\int_{\overline{X}_{(1,\ell)}E_\ell N_{\ell-1}\backslash G_\ell}\int_{F_{\ell-1}}W(e_\ell\widetilde{y}g)$$

$$\int_{\overline{X}^{(\ell,n)}\times Z}(\widetilde{F_{\tilde\phi}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(\widetilde{y}g),w_{1,n}^{-1}z\begin{pmatrix}I_{\ell+1}&&\\ &(-1)^{n-1}&\\ &&I_{r-1}\end{pmatrix}\right.$$

$$\left.\cdot\begin{pmatrix}1&0&0&-y'v'\\ &I_{\ell-1}&0&-yv_1'\\ &&1&0\\ &&&I_r\end{pmatrix},I_{n-1}\right)\cdot\psi_a^{(-1)^{n-1}}(\overline{x})dzd\overline{x}dydg$$

$$=\int_{\overline{X}_{(1,\ell)}E_\ell N_{\ell-1}\backslash G_\ell}\int_{F_{\ell-1}}W(e_\ell\widetilde{y}g)\int_{\overline{X}^{(\ell,n)}\times Z}(\widetilde{F_{\tilde\phi}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(\widetilde{y}g),w_{1,n}^{-1}z\right.$$

$$\left.\begin{pmatrix}I_{\ell+1}&&\\ &(-1)^{n-1}&\\ &&I_{r-n}\end{pmatrix},I_{n-1}\right)\cdot\psi_a^{(-1)^{n-1}}(\overline{x})dzd\overline{x}dydg.$$

For the last equality, if $z\equiv\begin{pmatrix}1&0&z\\ &I_\ell&0\\ &&I_r\end{pmatrix}$, we change variable $z\mapsto z+y'v'$.

Note also that $w_{1,n}^{-1}\begin{pmatrix}1&&&\\ &I_{\ell-1}&0&yv_1'\\ &&1&0\\ &&&I_r\end{pmatrix}w_{1,n}=\begin{pmatrix}I_{\ell-1}&0&yv_1'&\\ &1&0&\\ &&I_r&\\ &&&1\end{pmatrix}$. Now we go on to get

$$\int_{\overline{X}_{(1,\ell)}N_{\ell-1}\backslash G_\ell}W(e_\ell g)\int_{\overline{X}^{(\ell,n)}\times Z}(\widetilde{F_{\tilde\phi}})\left(\overline{x}\delta_{\ell,n}i_{\ell,n}(g),w_{1,n}^{-1}z\right.$$

$$\left.\begin{pmatrix}I_{\ell+1}&&\\ &(-1)^{n-1}&\\ &&I_{r-1}\end{pmatrix},I_{n-1}\right)$$

$$\cdot\psi_a^{(-1)^{n-1}}(\overline{x})dzd\overline{x}dg.$$

Note that $e_\ell\overline{X}_{(1,\ell)}e_\ell^{-1}=E_\ell$. Changing g to $e_\ell g$ we get (3.22). ∎

It remains to show (in the domain (3.21)) that the l.h.s. of (3.20) as written in (3.22) is actually $\widetilde{A}(W, \varphi_\phi)$. This we do now. Note that in the domain (3.24) all the following manipulations in the integral (3.22) are justified.

LEMMA 3.12: *We have*

$$\widetilde{F}_{\tilde{\phi}}(m(w_{1,n}^{-1})h, I_n, b) = F_{M(\phi)}((m(w_{1,n}^{-1})h)^{\omega_n^{n-1}}, I_n, b_{\ell,n-1}^* b^*), \tag{3.26}$$

where $M(\phi)(h, I_{2n-2}, b) = \int_{\overline{U}_n} \phi(\bar{u}w_n^{-1}h, I_{2n-2}, b)d\bar{u}$.

Proof: The l.h.s. of (3.26) is, by (3.13),

$$\int_{F^{n-1}} \int_{\overline{U}_{n-1}} \phi\left(m\begin{pmatrix} 1 & \\ y & I_{n-1}\end{pmatrix}^{\omega_n^{n-1}} h^{\omega_n^{n-1}}, \bar{u}w_{n-1}^{-1}, b_{\ell,n-1}^* b^*\right) d\bar{u}dy$$

$$= \int_{\overline{U}_n} \phi\left(\bar{u}\begin{pmatrix} 1 & & \\ & w_{n-1}^{-1} & \\ & & 1\end{pmatrix} h^{\omega_n^{n-1}}, I_{2n-2}, b_{\ell,n-1}^* b^*\right) d\bar{u}$$

$$= F_{M(\phi)}\left(w_n\begin{pmatrix} 1 & & \\ & w_{n-1}^{-1} & \\ & & 1\end{pmatrix} h^{\omega_n^{n-1}}, I_{2n-2} b_{\ell,n-1}^* b^*\right)$$

$$= F_{M(\phi)}\left(\left(m(w_{1,n}^{-1})h\right)^{\omega_n^{n-1}}, I_n, b_{\ell,n-1}^* b^*\right). \quad ∎$$

Thus, (3.22) equals

$$\int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}\times Z}$$
$$F_{M(\phi)}\left[\left(m(w_{1,n}^{-1}z\cdot\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1}\end{pmatrix})\bar{x}\gamma'_{\ell,n}\right)^{\omega_n^{n-1}}\right.$$
$$\left. i_{\ell,n}(g), I_n, b_{\ell,n-1}^*\right] \cdot \psi_a^{(-1)^{n-1}}(\bar{x})dzd\bar{x}dy \tag{3.27}$$

where $\gamma'_{\ell,n} = \delta_{\ell,n} \cdot i_{\ell,n}(e_\ell)$.

Note that conjugation of $\bar{x}$ by $m\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1}\end{pmatrix}$ changes $\psi_a^{(-1)^{n-1}}(\bar{x})$ to $\psi_a(\bar{x})$. Perform the conjugation by $m(w_{1,n}^{-1})$. We get, writing

$\overline{x} = \overline{u}\begin{pmatrix} v_1 & v & y \\ 0 & 0 & v' \\ 0 & 0 & v_1' \end{pmatrix}$, where $v_1 \in M_{r\times\ell}(F), v \in M_{r\times\ell}(F)$,

$$\int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int F_{M(\phi)}\Bigg[\Bigg(\overline{u}\begin{pmatrix} 0 & v_1' & 0 \\ v & y & v_1 \\ 0 & v' & 0 \end{pmatrix}$$

$$\cdot\, m\begin{pmatrix} I_\ell & & \\ & I_r & \\ & z & 1 \end{pmatrix}\gamma_{\ell,n}\Bigg)^{\omega_n^{n-1}} i_{\ell,n}(g), I_n, b^*_{\ell,n-1}\Bigg]\psi_a(\overline{x})d(\cdots) \tag{3.28}$$

where

$$\gamma_{\ell,n} = m\Bigg(w_{1,n}^{-1}\cdot\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1} \end{pmatrix}\Bigg)\cdot\gamma'_{\ell,n}. \tag{3.29}$$

If n is odd then $\overline{u}\begin{pmatrix} 0 & v_1' & 0 \\ 0 & 0 & v_1 \\ 0 & 0 & 0 \end{pmatrix}$ lies in the Levi part of $P_n^{\omega_n}$ and

$$F_{M(\phi)}\Bigg(\overline{u}\begin{pmatrix} 0 & v_1' & 0 \\ 0 & 0 & v_1 \\ 0 & 0 & 0 \end{pmatrix} h, I_n, b\Bigg) = F_{M(\phi)}\Bigg(h, \begin{pmatrix} 1 & & \\ v_1 & I_r & \\ 0 & 0 & I_\ell \end{pmatrix}, b\Bigg).$$

Note that in this case,

$$m\begin{pmatrix} I_\ell & & \\ & I_r & \\ & z & 1 \end{pmatrix} = \overline{u}\begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z' \\ 0 & 0 & 0 \end{pmatrix}^{\omega_n}.$$

If n is even, then

$$\overline{u}\begin{pmatrix} 0 & v_1' & 0 \\ 0 & 0 & v_1 \\ 0 & 0 & 0 \end{pmatrix}^{\omega_n} = m\begin{pmatrix} I_\ell & & \\ & I_r & \\ & v_1' & 1 \end{pmatrix}$$

and

$$F_{M(\phi)}\Bigg(\Bigg(\overline{u}\begin{pmatrix} 0 & v_1' & 0 \\ 0 & 0 & v_1 \\ 0 & 0 & 0 \end{pmatrix}\Bigg)^{\omega_n} h, I_n, b\Bigg) = F_{M(\phi)}\Bigg(h, \begin{pmatrix} 1 & & \\ v_1 & I_r & \\ 0 & 0 & I_\ell \end{pmatrix}, b\Bigg).$$

Thus (3.28) equals, putting $\mu_{\ell,n} = \gamma_{\ell,n}^{\omega_n^{n-1}}$,

$$\int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int F_{M(\phi)}\Bigg(\Bigg(\overline{u}\begin{pmatrix} 0 & z & 0 \\ v & y & z' \\ 0 & v' & 0 \end{pmatrix}\Bigg)^{\omega_n^n} \mu_{\ell,n} i_{\ell,n}(g),$$

$$\begin{pmatrix} 1 & & \\ v_1 & I_r & \\ 0 & 0 & I_\ell \end{pmatrix}, b^*_{\ell,n-1}\Bigg)\psi(v_{r,\ell})d(\cdots) = \int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int$$

$$F_{M(\phi)}\left(m(w_{1,n})\cdot \overline{u}\begin{pmatrix} 0 & z & 0 \\ v & y & z' \\ 0 & v' & 0 \end{pmatrix}^{\omega_n^n} \mu_{\ell,n} i_{\ell,n}(g), w_{1,n}\right.$$

$$\left.\begin{pmatrix} I_r & 0 & v_1 \\ & I_\ell & 0 \\ & & 1 \end{pmatrix}, b^*_{\ell,n-1}\right)\psi(v_{r,\ell})d(\cdots) = \int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} \int$$

$$F_{M(\phi)}\left(\left(\overline{x} m(w_{1,n})\right)^{\omega_n^n} \mu_{\ell,n} i_{\ell,n}(g), w_{1,n}\begin{pmatrix} I_r & 0 & v_1 \\ & I_\ell & 0 \\ & & 1 \end{pmatrix}, b^*_{\ell,n-1}\right)$$

$$\psi_a(\overline{x}) d(v_1,\overline{x}) dg = \int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} \int F_{M(\phi)}\Bigg(\overline{x}^{\omega_n^n} \widetilde{\mu}_{\ell,n} i_{\ell,n}(g), w_{1,n}$$

$$\begin{pmatrix} I_r & 0 & v_1 \\ & I_\ell & 0 \\ & & 1 \end{pmatrix}, b^*_{\ell,n-1}\Bigg)\psi_a(\overline{x}) d(v_1,\overline{x}) dg, \tag{3.30}$$

where $\widetilde{\mu}_{\ell,n} = m(w_{1,n})^{\omega_n^n}\mu_{\ell,n}$. We have

$$\widetilde{\mu}_{\ell,n} = m\begin{pmatrix} 1 & & \\ & -I_\ell & \\ & & I_r \end{pmatrix}(\eta_{\ell,n} m(\varepsilon_{\ell,n}))^{\omega_n}, \quad \text{if } n \text{ is odd}$$

(3.31)

$$\widetilde{\mu}_{\ell,n} = m\begin{pmatrix} 1 & & \\ & -I_{\ell+1} & \\ & & I_{r-1} \end{pmatrix}\beta_{\ell,n}, \qquad \text{if } n \text{ is even.}$$

Assume that n is odd. Then (3.30) equals

$$\int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} \int F_{M(\phi)}\Bigg(\left(\overline{x}\eta_{\ell,n} m(\varepsilon_{\ell,n})\right)^{\omega_n} i_{\ell,n}(g),$$

$$\begin{pmatrix} 1 & \\ & b^*_{\ell,n-1} \end{pmatrix} w_{1,n}\begin{pmatrix} I_r & 0 & v_1 \\ & I_\ell & 0 \\ & & 1 \end{pmatrix}\begin{pmatrix} I_r & & \\ & -I_\ell & \\ & & 1 \end{pmatrix}, I_{n-1}\Bigg)$$

$$\psi_a^{-1}(\overline{x}) dv_1,\overline{x}) dy = \int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n)}} \int F_{M(\phi)}\Bigg(\left(\overline{x}\eta_{\ell,n} m(\varepsilon_{\ell,n})\right)^{\omega_n}$$

$$i_{\ell,n}(g), w_{1,n}\begin{pmatrix} I_r & 0 & v_1 \\ & I_\ell & 0 \\ & & 1 \end{pmatrix} b^*_{\ell,n}, I_{n-1}\Bigg)\psi_a^{-1}(\overline{x}) d(v_1,\overline{x}) dg. \tag{3.32}$$

Now factor integration through $E_\ell N_{\ell-1}\backslash N_\ell$. Note that $i_{\ell,n}(g)$ commutes with ω_n, $m(\varepsilon_{\ell,n})$, $\eta_{\ell,n}$. We get (see [S1], p. 56) that (3.32) equals

$$\int_{N_\ell\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}}\int F_{M(\phi)}\Big((\overline{x}\eta_{\ell,n}m(\varepsilon_{\ell,n}))^{\omega_n}i_{\ell,n}(g), w_{1,n}$$

$$\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix} b^*_{\ell,n}, I_{n-1}\Big)\psi(z_{n-1})\cdot\psi_a^{-1}(\overline{x})dzd\overline{x}dy = \widetilde{A}(W,\varphi_\phi). \tag{3.33}$$

This completes the proof of Theorem 2 in case n is odd (and $r\geq 1$). Assume that n is even. Then (3.30) equals

$$\int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}}\int F_{M(\phi)}\Big(\overline{x}\beta_{\ell,n}i_{\ell,n}(g), \begin{pmatrix} 1 & \\ & b^*_{\ell,n-1}\end{pmatrix} w_{1,n}$$

$$\begin{pmatrix} I_r & 0 & v_1 \\ & I_\ell & 0 \\ & & 1\end{pmatrix}\begin{pmatrix} I_{r-1} & & \\ & -I_{\ell+1} & \\ & & 1\end{pmatrix}, I_{n-1}\Big)\psi_a(\overline{x})d(v_1,\overline{x})dg$$

$$= \int_{E_\ell N_{\ell-1}\backslash G_\ell}\int_{\overline{X}^{(\ell,n)}}\int F_{M(\phi)}\Big(\overline{x}\beta_{\ell,n}i_{\ell,n}(g), w_{1,n}$$

$$\begin{pmatrix} I_r & 0 & v_1 \\ & I_\ell & 0 \\ & & 1\end{pmatrix} b^*_{\ell,n}, I_{n-1}\Big)\psi_a(\overline{x})d(v_1,\overline{x})dg.$$

Now factor, as before, integration through $E_\ell N_{\ell-1}\backslash N_\ell$ to get

$$\int_{N_\ell\backslash G_\ell} W(g)\int_{\overline{X}^{(\ell,n)}}\int F_{M(\phi)}\Big(\overline{x}\beta_{\ell,n}i_{\ell,n}(g), w_{1,n}\begin{pmatrix} I_{n-1} & z \\ 0 & 1\end{pmatrix} b^*_{\ell,n}, I_{n-1}\Big)$$

$$\psi_a(\overline{x})dzd\overline{x}dg = \widetilde{A}(w,\varphi_\phi). \tag{3.34}$$

This completes the proof of Theorem 2, in case $r\geq 1$. ∎

4. Proof of Theorem 2 in case $r = n-\ell-1 = 0$

Assume that $\ell = n-1$. We omit (in Section 6) the technical justifications as they are easy repetitions of those needed for Section 3.

a. Direct Substitution of φ_ϕ in $A(W,\varphi_\phi)$. This is done as in the previous case. We get (in a domain D of the form (3.3))

$$A(W,\varphi_\phi) = \int_{N_\ell\backslash G_\ell} W(g)\int_{F^{n-1}}\phi\Big(m(w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix})i_{\ell,n}(g),$$

$$I_{2n-2}, I_{n-1}\Big)\psi(z_{n-1})dzdg. \tag{4.1}$$

b. LEMMA 4.1: *We have (in D)*

$$A(W,\varphi_\phi)=\int_{V_\ell\backslash G_\ell}W(g)\phi(m(w_{1,n})i_{\ell,n}(g),I_{2n-2},I_{n-1})dg \tag{4.2}$$

where

$$V_\ell=\left\{\begin{pmatrix} e & 0 & y\\ & 1 & 0\\ & & e^*\end{pmatrix}\in G_\ell\middle| e\in Z_\ell\right\}.$$

Proof: We have

$$\phi\left(m(w_{1,n})\begin{pmatrix} I_{n-1} & z\\ & 1\end{pmatrix})h,I_{2n-2},I_{n-1}\right)=$$
$$\phi(u_z\cdot m(w_{1,n})\begin{pmatrix} I_{n-1} & z\\ & 1\end{pmatrix})h,I_{2n-2},I_{n-1}$$

where $u_z=m(w_{1,n})u\begin{pmatrix} z & 0\\ 0 & z'\end{pmatrix}m(w_{1,n})^{-1}$, and this gives

$$\phi\left(m(w_{1,n})i_{\ell,n}\begin{pmatrix} I_{n-1} & z & *\\ & 1\, . & z'\\ & & I_{n-1}\end{pmatrix}h,I_{2n-2},I_{n-1}\right).$$

Using this in (4.1) we get

$$\begin{aligned}A(W,\varphi_\phi)&=\int_{N_\ell\backslash G_\ell}\int_{F^{n-1}}W\left(\begin{pmatrix} I_{n-1} & z & A_z\\ & 1 & z'\\ & & I_{n-1}\end{pmatrix}g\right)\phi(m(w_{1,n})i_{\ell,n}\\ &\qquad\left(\begin{pmatrix} I_{n-1} & z & A_z\\ & 1 & z'\\ & & I_{n-1}\end{pmatrix}g\right)\\ &=\int_{V_\ell\backslash G_\ell}W(g)\phi\big(m(w_{1,n})i_{\ell,n}(g),I_{2n-2},I_{n-1}\big)dg.\end{aligned}$$

(A_z is such that $\begin{pmatrix} I_{n-1} & z & A_z\\ & 1 & z'\\ & & I_{n-1}\end{pmatrix}\in\mathrm{SO}_{2n-1}$.) ∎

c. FACTORING INTEGRATION THROUGH H_ℓ. Since $\ell=n-1$, we can embed $H_\ell=\mathrm{SO}_{2n-2}(F)$ in $G_\ell=\mathrm{SO}_{2n-1}(F)$ by

$$j_{n-1,\ell}\begin{pmatrix} A & B\\ C & D\end{pmatrix}=\begin{pmatrix} A & & B\\ & 1 & \\ C & & D\end{pmatrix}.$$

Note that $V_\ell \subset H_\ell$ is indeed the standard unipotent subgroup of H_ℓ and that $\begin{pmatrix} A & & B \\ & 1 & \\ C & & D \end{pmatrix}$ further embeds as $\begin{pmatrix} A & & B \\ & I_2 & \\ C & & D \end{pmatrix}$ in $H_n = \mathrm{SO}_{2n}(F)$, where conjugation by $m(w_{1,n})$ takes it to $\begin{pmatrix} 1 & & & \\ & A & B & \\ & C & D & \\ & & & 1 \end{pmatrix}$. Therefore, factoring integration in (4.2) through H_ℓ gives (in D)

(4.3)

$$A(W,\varphi_\phi) = \int_{H_\ell\backslash G_\ell}\int_{V_\ell\backslash H_\ell} W(j_{n-1,\ell}(h)g)\phi\Big(m(w_{1,n})i_{\ell,n}(g), h, I_{n-1}\Big)dhdg.$$

d. Applying the Functional Equation for $\pi\times\tau'$. The inner dh-integral in (4.3) is the local integral for $\pi\times\tau'$ on $\mathrm{SO}_{2n-1}\times\mathrm{GL}_{n-1}$. Let us apply the local functional equation, justifications being as in Section 3.c. We get (see Section 1)

$$\frac{\gamma(\pi\times\tau', s-\zeta,\psi)}{\gamma(\tau',\Lambda^2, 2(s-\zeta)-1,\psi)}A(W,\varphi_\phi) =$$

(4.4)

$$\int_{H_\ell\backslash G_\ell}\int_{V_\ell\backslash H_\ell} W(j_{n-1,\ell}(h)a_{n-1,\ell}^{n-1}g)\phi^\sim(m(w_{1,n})i_{\ell,n}(g), h, I_{n-1})dhdg.$$

Here, $\phi^\sim$ is defined similarly to (3.6). Put, for $h\in H_n$,

$$\phi_h(h',b) = \phi(h,h',b),\quad h'\in H_{n-1},\quad b\in \mathrm{GL}_{n-1}(F),$$

ϕ_h lies in $V_{\rho_{\tau',s-\zeta}}$. Then

$$\phi^\sim(h,h',b) = M(w_{n-1},\phi_h)\Big(h'^{\omega_{n-1}^{n-1}}, b_{n-1}^* b^*\Big).$$

We continue the calculation (in the domain of convergence D', of the form (3.7)) of the integral on the r.h.s. of (4.4).

e. Unfolding $B(W,\varphi_\phi)$ Back. Denote by $B(W,\varphi_\phi)$ the r.h.s. of (4.4). We have (in D')

$$B(W,\varphi_\phi) = \int_{H_\ell\backslash G_\ell}\int_{V_\ell\backslash H_\ell} W(j_{n-1,\ell}(h)a_{n-1,\ell}^{n-1}g)\widetilde{\phi}\Big(\begin{pmatrix} 1 & & \\ & h & \\ & & 1\end{pmatrix}$$

(4.5)

$$m(w_{1,n})^{\omega_n^{n-1}} i_{\ell,n}(g), I_{2n-2}, I_{n-1}\Big)dhdg$$

where $\widetilde{\phi}(h,h',b) = \phi^{\sim}(h^{\omega_n^{n-1}}, h'b)$, and then we get

$$B(W,\varphi_\phi) = \int_{V_\ell\backslash G_\ell} W(a_{n-1,\ell}^{n-1}g)\widetilde{\phi}(m(w_{1,n})^{\omega_n^{n-1}} i_{\ell n}(g), I_{2n-2}, I_{n-1}\Big)dg$$

$$= \int_{N_\ell\backslash G_\ell}\int_{F^{n-1}} W(a_{n-1,\ell}^{n-1}g)\widetilde{\phi}(m(w_{1,n}\begin{pmatrix} I_{n-1} & z \\ 0 & 1\end{pmatrix}) \cdot \delta_1^{n-1} \cdot i_{\ell,n}(g),$$

$$I_{2n-2}, I_{n-1})\psi^{(-1)^{n-1}}(z_{n-1})dzdg \tag{4.6}$$

where

$$\delta_1 = \begin{pmatrix} I_{n-2} & & & & \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \\ & & & & I_{n-2}\end{pmatrix}.$$

f. Application of Shahidi's Functional Equation for $\tau' \times \mu$. From (4.6), we have

$$B(W,\varphi_\phi) = \int_{N_\ell\backslash G_\ell} W(a_{n-1,\ell}^{n-1}g)\int_{F^{n-1}} F_{\widetilde{\phi}}\Big(\delta_1^{n-1} i_{\ell,n}(g), w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix}, I_{n-1}\Big)$$
$$\psi^{(-1)^{n-1}}(z_{n-1})dzdg.$$

As in Section 3.e, we have (see (3.12), (3.13))

$$B(W,\varphi_\phi) = c_\psi\big(\mu_{s+\zeta+(n-1)/2} \times (\tau')^*_{-(s-\zeta)+n/2}\big) \cdot$$
$$\cdot \int_{N_\ell\backslash G_\ell} W(a_{n-1,\ell}^{n-1}g)\int_{F^{n-1}} \widetilde{F_{\widetilde{\phi}}}\Big(\delta_1^{n-1} i_{\ell,n}(g), w_{1,n}^{-1}\begin{pmatrix} 1 & t \\ & I_{n-1}\end{pmatrix}$$
$$\begin{pmatrix} I_{n-1} & \\ & (-1)^{n-1}\end{pmatrix}, I_{n-1}\Big)\psi(t_1)dtdg. \tag{4.7}$$

Reasoning as before, we continue the calculation in a domain D'' of the form (3.14). Denote the integral in (4.7) by $C(W,\phi)$. Using (3.11), we have

$$\frac{\gamma(\pi \times \tau', s-\zeta, \psi)}{\gamma(\tau, \Lambda^2, 2(s-\zeta)-1, \psi)} A(W,\varphi_\phi) = C(W,\phi). \tag{4.8}$$

It is easy to check that in both cases (n even or odd), we have

$$C(W,\phi) = \int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g)\widetilde{F_{\widetilde{\phi}}}(m(w_{1,n}^{-1})i_{\ell,n}(g), I_n, I_{n-1})dg \tag{4.9}$$

(see (3.15) for notation).

g. Factoring Integration Through $\begin{pmatrix} * & \\ * & I_{\ell-1} \end{pmatrix}^{\wedge}$. This gives

$$C(W,\phi)=\int_{\hat{C}_\ell E_\ell N_{\ell-1}}\int_{F^*}\int_{\overline{X}_{(1,\ell)}} W\left(\overline{y}j_{1,\ell}\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix} g\right)\mu(t)|t|^{s+\zeta-\frac{1}{2}}$$

$$\widetilde{F_{\tilde\phi}}\left(m(w_{1,n}^{-1})i_{\ell,n}(g),I_n,I_{n-1}\right)dyd^*tdg. \tag{4.10}$$

h. Application of the Functional Equation for $\pi\times\mu$.

$$\frac{\gamma(\pi\times\tau',s-\zeta,\psi)\gamma(\pi\times\mu,s+\zeta,\psi)}{\gamma(\tau,\Lambda^2,2(s-\zeta)+1,\psi)}A(W,\varphi_\phi)=$$

$$\int_{\hat{C}_\ell E_\ell N_{\ell-1}\backslash G_\ell}\int_{F^*}\int_{\overline{X}_{(1,\ell)}} W(\hat{c}_{1,\ell}\overline{y}j_{1,\ell}\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}a_{1,\ell})\mu^{-1}(t)|t|^{\frac{1}{2}-(s+\zeta)}.$$

$$\cdot\widetilde{F_{\tilde\phi}}(m(w_{1,n}^{-1})i_{\ell,n}(g),I_n,I_{n-1})dyd^*tdg. \tag{4.11}$$

We continue, as before, in a domain D''' of the form (3.21). Denote the integral in (4.11) by $D(W,\phi)$.

i. Unfolding $D(W,\phi)$ Back. Note that conjugation by $a_{1,\ell}$ flips t to t^{-1} and takes

$$\overline{y}=\begin{pmatrix} 1 & \\ y & I_{\ell-1}\end{pmatrix}^{\wedge}\quad\text{to}\quad\begin{pmatrix} 1 & 0 & 0 & y' & 0\\ & I_{\ell-1} & 0 & 0 & y\\ & & 1 & 0 & 0\\ & & & I_{\ell-1} & 0\\ & & & & 1\end{pmatrix}\equiv e$$

which is a general element of E_ℓ, and clearly $g\mapsto\widetilde{F_{\tilde\phi}}(m(w_{1,n}^{-1})i_{\ell,n}(g),I_n,I_{n-1})$ is left invariant by e. Thus

$$\begin{aligned}D(W,(\phi)&=\int_{\overline{X}_{(1,\ell)}E_\ell N_{\ell-1}\backslash G_\ell}\int_{\overline{X}_{(1,\ell)}} W(\hat{c}_{1,\ell}a_{1,\ell}\overline{y}g)\\ &\qquad\widetilde{F_{\tilde\phi}}(m(w_{1,n}^{-1})i_{\ell,n}(g),I_n,I_{n-1})dydg\\ &=\int_{\overline{X}_{(1,\ell)}E_\ell N_{\ell-1}\backslash G_\ell}\int_{E_\ell} W(e_\ell eg)\widetilde{F_{\tilde\phi}}(m(w_{1,n}^{-1})i_{\ell,n}(eg),I_n,I_{n-1})dedg\\ &\qquad\left(e_\ell=\hat{c}_{1,\ell}a_{1,\ell}=\begin{pmatrix} & & 1\\ & -I_{2\ell-1} & \\ 1 & & \end{pmatrix}\right)\\ &=\int_{\overline{X}_{(1,\ell)}N_{\ell-1}\backslash G_\ell} W(e_\ell g)\widetilde{F_{\tilde\phi}}(m(w_{1,n}^{-1})i_{\ell,n}(g),I_n,I_{n-1})dg\\ &=\int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g)\widetilde{F_{\tilde\phi}}(m(w_{1,n}^{-1})i_{\ell,n}(e_\ell g),I_n,I_{n-1})dg\end{aligned}$$

$$\overset{\text{as in (3.26)}}{=} \int_{E_\ell N_{\ell-1}\backslash G_\ell} W(g)F_{M(\phi)}\left(m(w_{1,n}^{-1})^{\omega_n^{n-1}} i_{\ell,n}(e_\ell g), I_n, b_{n-1}^*\right)dg.$$

We have

$$m(w_{1,n}^{-1})^{\omega_n^{n-1}} i_{\ell,n}(e_\ell) = m\begin{pmatrix} & -I_{n-1} \\ 1 & \end{pmatrix}^{\omega_n^n}.$$

Factoring integration through $E_\ell N_{\ell-1}\backslash N_\ell$, we get

$$\begin{aligned} D(W,\phi) &= \int_{N_\ell\backslash G_\ell} W(g)\int F_{M(\phi)}\left(\left(m(w_{1,n}^{-1})\begin{pmatrix} 1 & t \\ & I_{n-1}\end{pmatrix}\right)^{\omega_n^n} i_{\ell,n}(g),\right. \\ &\qquad \left.\begin{pmatrix} 1 & \\ & -b_{n-1}^*\end{pmatrix}, I_{n-1}\right)\cdot \psi(t_1)dtdg \\ &= \int_{N_\ell\backslash G_\ell} W(g)\int F_{M(\phi)}(i_{\ell,n}(g), \begin{pmatrix} 1 & \\ & -b_{n-1}^*\end{pmatrix} w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix}, \\ &\qquad I_{n-1})\psi^{-1}(z_{n-1})dzdg. \end{aligned}$$

We have $w_{1,n}^{-1}\begin{pmatrix} 1 & \\ & -b_{n-1}^*\end{pmatrix} w_{1,n} = b_n^*$ and hence

$$\begin{aligned} D(W,\phi) &= \int_{N_\ell\backslash G_\ell} W(g)\int F_{M(\phi)}\left(i_{\ell,n}(g), w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix} b_n^*, I_{n-1}\right) \\ &\qquad \psi(z_{n-1})dzdg \\ &= \widetilde{A}(W,\varphi_\phi). \end{aligned}$$

This proves Theorem 2 in case $r=0$.

5. Proof of Theorem 2 in case $r = n-\ell-1 < 0$

Assume that $\ell \geq n$ (i.e. $r<0$). We give the details briefly. The technical justifications are similar in nature to those of Section 3 and even easier, so we omit them.

Substitute φ_ϕ in $A(W,\varphi_\phi)$ to get

$$A(W,\varphi_\phi) = \int_{V_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\bar{x} j_{n,\ell}(h))\int_{F_{n-1}}$$

$$\phi\left(m(w_{1,n}\begin{pmatrix} I_{n-1} & z \\ 0 & 1\end{pmatrix})h, I_{2n-2}, I_{n-1}\right)\psi(z_{n-1})dzd\bar{x}dh. \tag{5.1}$$

This integral converges absolutely in a domain D of type (3.3).

We have

$$\psi(z_{n-1})W(\overline{x}j_{n,\ell}(h)) = W(\overline{x}'j_{n,\ell}\left(m\begin{pmatrix} I_{n-1} & z \\ 0 & 1\end{pmatrix}h\right))$$

with $d\overline{x} = d\overline{x}'$, and so factoring the dz-integration in (5.1) it becomes (in the domain D)

$$\int_{m(Z_{n-1})U_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}j_{n,\ell}(h))\phi(m(w_{1,n})h, I_{2n-2,I_{n-1}})d\overline{x}dh \tag{5.2}$$

where $m(Z_{n-1}) = \left\{m\begin{pmatrix} z & 0 \\ 0 & 1\end{pmatrix} \middle| z \in Z_{n-1}\right\}$. Factor the dh-integration in (5.2) through $\left\{m\begin{pmatrix} I_{n-1} & \\ y & 1\end{pmatrix} \middle| y \in F^{n-1}\right\}$, noting that $h \mapsto \phi(m(w_{1,n})h, I_{2n-2}, I_{n-1})$ is left-$m\begin{pmatrix} I_{n-1} & \\ y & 1\end{pmatrix}$ invariant. We get

$$\int_{\widetilde{Z}_nU_n\backslash H_n}\int W\left(\begin{pmatrix} I_{n-1} & & \\ y & 1 & \\ x & r & I_{\ell-n}\end{pmatrix}^{\wedge}\cdot j_{n,\ell}(h)\right)\phi(m(w_{1,n})$$

$$h, I_{2n-2}, I_{n-1})d(y,x,r)dh. \tag{5.3}$$

Here $\widetilde{Z}_n = m\left(Z_{n-1}\cdot\left\{\begin{pmatrix} I_{n-1} & \\ y & 1\end{pmatrix} \middle| y \in F^{n-1}\right\}\right)$.

Now factor integration in (5.3) through

$$H_{n-1} = \left\{\begin{pmatrix} A & & B \\ & I_2 & \\ C & & D\end{pmatrix} \middle| \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in H_{n-1}\right\}.$$

We get

$$\int_{\widetilde{Z}_nU_nH_{n-1}\backslash H_n}\int_{F_{\ell-n}}\int_{V_{n-1}\backslash H_{n-1}}\int_{\overline{X}_{(n-1,\ell)}}$$

$$W\left(\overline{x}j_{n-1,\ell}(h')\begin{pmatrix} I_{n-1} & & \\ & 1 & \\ & r & I_{\ell-n}\end{pmatrix}^{\wedge} j_{n,\ell}(h)\right)\cdot$$

$$\cdot\,\phi(m(w_{1,n})h, h', I_{n-1})d\overline{x}dh'drdh. \tag{5.4}$$

Note that $\widetilde{Z}_nU_nH_{n-1}$ is a subgroup of H_n. Now apply the functional equation for $\pi \times \tau'$ (on $\mathrm{SO}_{2\ell+1} \times \mathrm{GL}_{n-1}$),

$$\frac{\gamma(\pi\times\tau', s-\zeta, \psi)}{\gamma(\tau', \Lambda^2, 2(s-\zeta)-1, \psi)}\quad A(W, \varphi_\phi)$$

$$= \int_{\widetilde{Z}_n U_n H_{n-1}\backslash H_n} \int_{F_{\ell-n}} \int_{V_{n-1}\backslash H_{n-1}} \int_{\overline{X}_{(n-1,\ell)}} W\left(\hat{c}^{n-1}_{n-1,\ell}\overline{x} j_{n-1,\ell}(h') a^{n-1}_{n-1,\ell} \begin{pmatrix} I_{n-1} & & \\ & 1 & \\ & r & I_{\ell-n} \end{pmatrix}^{\wedge} j_{n,\ell}(h)\right) \cdot$$

$$\cdot\, \phi^{\sim}(m(w_{1,n})h, h', I_{n-1}) d\overline{x} dh' dr dh. \tag{5.5}$$

The last integral converges in a domain D' of the form

$$-\operatorname{Re}(\zeta) + A \le \operatorname{Re}(s) \le \operatorname{Re}(\zeta) + B \quad (A \gg 0, B \ll 0)$$

where A, B depend on π, τ' and μ and (5.5) is understood in the sense of analytic continuation (equality of rational functions of q^{-s}). Here

$$\phi^{\sim}(h, h', b) = M(w_{n-1}, \phi_h)\left(h'^{\omega^{n-1}_{n-1}}, b^*_{n-1} b^*\right)$$

for $h \in H_n$, $h' \in H_{n-1}$, $b \in \mathrm{GL}_{n-1}(F)$ and $\phi_h(h', b) = \phi(h, h', b)$. We continue in the domain D'.

Note that

$$j_{n-1,\ell}(h') a^{n-1}_{n-1,\ell} = a^{n-1}_{n-1,\ell} j_{n-1,\ell}\left(h'^{\omega^{n-1}_{n-1}}\right)$$

and

$$\phi^{\sim}(m(w_{1,n})h, h', I_{n-1}) = \phi^{\sim}\left(\begin{pmatrix} 1 & & \\ & h'^{\omega^{n-1}_{n-1}} & \\ & & 1 \end{pmatrix} m(w_{1,n})h, I_{2n-2}, I_{n-1}\right).$$

Now the integral (5.5) becomes

$$\int_{\widetilde{Z}_n U_n\backslash H_n} \int_{F_{\ell-n}} \int_{\overline{X}_{(n-1,\ell)}} W\left(\hat{c}^{n-1}_{n-1,\ell}\overline{x} a^{n-1}_{n-1,\ell} \begin{pmatrix} I_{n-1} & & \\ & 1 & \\ & r & I_{\ell-n} \end{pmatrix}^{\wedge} j_{n,\ell}(h)\right) \cdot$$

$$\cdot\, \phi^{\sim}(m(w_{1,n})h, I_{2n-2}, I_{n-1}) d\overline{x} dr dh \tag{5.6}$$

$$= \int_{m(Z_{n-1})U_n\backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\overline{x} d^{n-1}_{n,\ell} j_{n,\ell}(h)) \phi^{\sim}(m(w_{1,n})h, I_{2n-2}, I_{n-1}) d\overline{x} dh.$$

Here $d_{n,\ell} = \hat{c}_{n-1,\ell} a_{n-1,\ell} = \begin{pmatrix} I_{n-2} & & & & \\ & 0 & & 1 & \\ & & -I_{2(\ell-n)+3} & & \\ & 1 & & 0 & \\ & & & & I_{n-2} \end{pmatrix}$. Rewrite (5.6) as ($\widetilde{\phi}$ defined as before)

$$\int_{m(Z_{n-1})U_n\backslash H_n} \int_{\overline{X}_{(n,\ell)}} W\left(\overline{x} d^{n-1}_{\ell,n} j_{n,\ell}(h)\right) \widetilde{\phi}\left(\left(m(w_{1,n})h\right)^{\omega^{n-1}_n}, I_{2n-2}, I_{n-1}\right) d\overline{x} dh. \tag{5.7}$$

Factor integration through $\left\{m\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix}\right\}$ in case $n-1$ is even and through

$$\left\{\left(\begin{array}{ccc|ccc} I_{n-2} & 0 & z_1 & & & \\ & 1 & 0 & & & \\ & & 1 & & & \\ \hline 0 & -z_2 & 0 & 1 & 0 & z_1' \\ 0 & 0 & z_2 & & 1 & 0 \\ 0 & 0 & 0 & & & I_{n-2}\end{array}\right)\right\}$$

in case $n-1$ is odd. Let

$$V_n^{(n)} = \begin{cases} V_n, & n \text{ odd}, \\ m(Z_{n-1})U_n\left\{\left(\begin{array}{ccc|ccc} I_{n-2} & 0 & z_1 & & & \\ & 1 & 0 & & & \\ & & 1 & & & \\ \hline 0 & -z_2 & 0 & 1 & 0 & z_1' \\ 0 & 0 & z_2 & & 1 & 0 \\ 0 & 0 & 0 & & & I_{n-2}\end{array}\right)\right\}, & n \text{ even}.\end{cases}$$

Then (5.7) becomes

$$\int_{V_n^{(n)}\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W\left(\overline{x}d_{n,\ell}^{n-1}j_{n,\ell}(h)\right)\int_{F_{n-1}} F_{\tilde{\phi}}\Big(m(w_{1,n}^{-1})(m(w_{1,n})h)^{\omega_n^{n-1}}, w_{1,n}$$

$$\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix}, I_{n-1}\Big)\cdot\psi^{(-1)^{n-1}}(z_{n-1})dzd\overline{x}dh. \tag{5.8}$$

Now we are ready to apply Shahidi's functional equation. We get (interpretation and notation as before)

$$\frac{\gamma(\pi\times\tau', s-\zeta,\psi)}{\gamma(\tau,\Lambda^2,2(s-\zeta)-1,\psi)}A(W,\varphi_\phi) =$$

$$\int_{V_n^{(n)}\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}d_{n,\ell}^{n-1}j_{n,\ell}(h))\int_{F^{n-1}} \widetilde{F_{\tilde{\phi}}}(m(w_{1,n}^{-1})(m(w_{1,n})h)^{\omega_n^{n-1}}, w_{1,n}^{-1}$$

$$\begin{pmatrix} 1 & t \\ & I_{n-1}\end{pmatrix}\begin{pmatrix} I_{n-1} & \\ & (-1)^{n-1}\end{pmatrix}, I_{n-1})\cdot\psi(t_1)dtd\overline{x}dh. \tag{5.9}$$

The last integral converges absolutely in the domain D'' of the form

$$-\operatorname{Re}(\zeta)+B\le\operatorname{Re}(s)\le\operatorname{Re}(\zeta)+A,$$
$$\operatorname{Re}(s)\le C,$$

where A, B, C are constants which depend on π, τ and μ ($A, C \ll 0; B \gg 0$). In the last domain (5.9) equals

$$\int_{V_n^{(n)}\backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\overline{x} d_{n,\ell}^{n-1} j_{n,\ell}(h)) \int_{F^{n-1}} \widetilde{F_{\tilde{\phi}}}\Big(m\Big(w_{1,n}^{-1}\begin{pmatrix} 1 & t \\ & I_{n-1}\end{pmatrix}\Big)\Big)$$

$$\delta_2^{n-1} h^{\omega_n^{n-1}}, I_n, I_{n-1})\psi(t_1)dt d\overline{x} dh. \tag{5.10}$$

Here

$$\delta_2 = \begin{pmatrix} I_{n-2} & & & & & \\ & 0 & 0 & 0 & 1 & \\ & 0 & 0 & -1 & 0 & \\ & 0 & -1 & 0 & 0 & \\ & 1 & 0 & 0 & 0 & \\ & & & & & I_{n-2} \end{pmatrix}.$$

We have

(5.11)

$$d_{n,\ell}^{n-1} j_{n,\ell}\left(\left[\delta_2^{n-1} m \begin{pmatrix} 1 & t \\ & I_{n-1}\end{pmatrix} \delta_2^{n-1}\right]^{\omega_n^{n-1}}\right) \cdot d_{n,\ell}^{n-1} = \begin{pmatrix} 1 & t & \\ & I_{n-1} & \\ & & I_{\ell-n}\end{pmatrix}^{\wedge}.$$

Using (5.11), (5.10) becomes

$$\int_{V_n^n\backslash H_n} \int_{\overline{X}_{(n,\ell)}} W(\overline{x} d_{n,\ell}^{n-1} j_{n,\ell}(h)) \widetilde{F_{\tilde{\phi}}}\Big(m(w_{1,n}^{-1})\delta_2^{n-1} h^{\omega_n^{n-1}}, I_n, I_{n-1}\Big) d\overline{x} dh. \tag{5.12}$$

In case $n-1$ is even, $V_n^n = Z_{n-1}^{\vee} U_n$, where

$$Z_{n-1}^{\vee} = \left\{ m\begin{pmatrix} 1 & \\ & z\end{pmatrix} \,\middle|\, z \in Z_{n-1}\right\}.$$

In case $n-1$ is odd, V_n^n is obtained from $V_n^{(n)}$ by "deleting" the coordinates a, b, c in

$$\begin{pmatrix} 1 & a & 0 & b & 0 & 0 & c & 0 & 0 \\ & I_{n-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 & -c \\ & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & I_{2(\ell-n)+1} & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 & -b \\ & & & & & & 1 & 0 & 0 \\ & & & & & & & I_{n-3} & a' \\ & & & & & & & & 1 \end{pmatrix}.$$

Now we want to "prepare" (5.12) for the application of the functional equation for $\pi \times \mu$. We check that $h \mapsto \widetilde{F}_{\tilde{\phi}}(m(w_{1,n}^{-1})\delta_2^{n-1} h^{w_n^{n-1}}, I_n, I_{n-1})$ is left-invariant under

$$d_{n,\ell}^{n-1}\begin{pmatrix}1&&\\y&I_{n-1}&\\&&I_{\ell-n}\end{pmatrix}^{\wedge}d_{n,\ell}^{n-1},$$

which is equal to $\left[\delta_2^{n-1}j_{n,\ell}\left(\begin{pmatrix}1&\\y&I_{n-1}\end{pmatrix}\right)\delta_2^{n-1}\right]^{\omega_n^{n-1}}$. Thus, factoring integration in (5.12) through the last subgroup (call it $\overline{Y}$) gives

$$\int_{\overline{Y}V_n^n\backslash H_n}\int_{\overline{X}_{(n,\ell)}}\int_{F^{n-1}}W\left(\begin{pmatrix}1&&\\y&I_{n-1}&\\&&I_{\ell-n}\end{pmatrix}^{\wedge}\overline{x}d_{n,\ell}^{n-1}j_{n,\ell}(h)\right)\cdot$$
$$\cdot\widetilde{F_{\tilde{\phi}}}(m(w_{1,n}^{-1})\delta_2^{n-1}h^{\omega_n^{n-1}},I_n,I_{n-1})dyd\overline{x}dh$$
$$=\int_{T\overline{Y}V_n^n\backslash H_n}\int_{\overline{X}_{(n-1,\ell-1)}}\int_{F^*}\int_{\overline{X}_{(1,\ell)}}W\left(\overline{x}_1j_{1,\ell}\begin{pmatrix}t&\\&t^{-1}\end{pmatrix}\begin{pmatrix}1&&\\&\overline{x}_2&\\&&1\end{pmatrix}\right.$$
$$\left.d_{n,\ell}^{n-1}j_{n,\ell}(h)\right)\mu(t)|t|^{s+\zeta-\frac{1}{2}}$$
$$\cdot\widetilde{F_{\tilde{\phi}}}(m(w_{1,n}^{-1})\delta_2^{n-1}h^{\omega_n^{n-1}},I_n,I_{n-1})d\overline{x}_1d^*td\overline{x}_2dh. \tag{5.13}$$

Here

$$T=\left\{m\begin{pmatrix}t&\\&I_{n-1}\end{pmatrix}\Big|t\in F^*\right\}.$$

Apply the functional equation for $\pi\times\mu$ on the inner $d\overline{x}_1d^*t$ integral in (5.13). We get

$$\frac{\gamma(\pi\times\tau',s-\zeta,\psi)\gamma(\pi\times\mu,s+\zeta,\psi)}{\gamma(\tau,\Lambda^2,2(s-\zeta)-1,\psi)}\quad A(W,\varphi_\phi)=$$
$$\int_{T\overline{Y}V_n^n\backslash H_n}\int_{\overline{X}_{(n-1,\ell-1)}}\int_{F^*}\int_{\overline{X}_{(1,\ell)}}W\left(\hat{c}_{1,\ell}\overline{x}_1j_{1,\ell}\begin{pmatrix}t&\\&t^{-1}\end{pmatrix}a_{1,\ell}\right.$$
$$\left.\begin{pmatrix}1&&\\&\overline{x}_2&\\&&1\end{pmatrix}d_{n,\ell}^{n-1}j_{n,\ell}(h)\right)\mu^{-1}(t)|t|^{\frac{1}{2}-(s+\zeta)}$$
$$\cdot\widetilde{F_{\tilde{\phi}}}(m(w_{1,n}^{-1})\delta_2^{n-1}h^{\omega_n^{n-1}},I_n,I_{n-1})d\overline{x}_1d^*td\overline{x}_2dh. \tag{5.14}$$

The integral (5.14) converges in a domain D''' of the form (3.21). We continue in D'''. The integral in (5.14) then equals (unfolding the d^*t integration back in)

$$\int_{\overline{Y}V_n^n\backslash H_n}\int_{\overline{X}_{(n-1,\ell-1)}}\int_{\overline{X}_{(1,\ell)}}W\left(\overline{x}_1\begin{pmatrix}1&&\\&\overline{x}_2&\\&&1\end{pmatrix}\delta_4^nj_{n,\ell}(\delta_3^{n-1}h)\right)\widetilde{F_{\tilde{\phi}}}$$

(5.15) $$\left(m(w_{1,n}^{-1})\delta_2^{n-1}h^{\omega_n^{n-1}}, I_n, I_{n-1}\right)d\overline{x}_1 d\overline{x}_2 dh.$$

Here

$$\hat{c}_{1,\ell}a_{1,\ell}d_{n,\ell} = j_{n,\ell}(\delta_3),$$

$$\delta_3 = \begin{pmatrix} 0 &&&&&&& 1 \\ & -I_{n-3} &&&&&& \\ && 0 &&& -1 && \\ &&& 1 &&&& \\ &&&& 1 &&& \\ && -1 &&& 0 && \\ &&&&&& -I_{n-3} & \\ 1 &&&&&&& 0 \end{pmatrix}$$

and

$$\hat{c}_{1,\ell}a_{1,\ell} = \delta_4 = \begin{pmatrix} && 1 \\ & -I_{2\ell-1} & \\ 1 && \end{pmatrix}.$$

Let us separate at this point the cases $n-1$ even and $n-1$ odd. Assume that $n-1$ is even. Note that δ_4 conjugates $j_{n,\ell}\left(\begin{pmatrix} 1 & \\ y & I_{n-1} \end{pmatrix}\right)$ to

$$j_{n,\ell}\left(\begin{array}{cc|cc} 1 & & y' & 0 \\ & I_{n-1} & 0 & y \\ \hline & & I_{n-1} & \\ & & & 1 \end{array}\right),$$

and that we may write $\overline{x}_1 \cdot \begin{pmatrix} 1 && \\ & \overline{x}_2 & \\ && 1 \end{pmatrix}$ in the form $\overline{x}j_{n,\ell}\left(m\begin{pmatrix} 1 & \\ y & I_{n-1} \end{pmatrix}\right)$, where $\overline{x} \in \overline{X}_{(n,\ell)}$. Thus (5.15) becomes

(5.16) $$\int_{Z_{n-1}^{\vee}U_n'\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}\delta_4 j_{n,\ell}(h))\widetilde{F_{\tilde{\phi}}}\left(m(w_{1,n}^{-1})h, I_n, I_{n-1}\right)d\overline{x}dh$$

where U_n' is the conjugate of U_n by $\begin{pmatrix} && 1 \\ & I_{2n-2} & \\ 1 && \end{pmatrix}$. In (5.16) change $h \mapsto h^{\omega_n}$, to obtain

(5.17)
$$\int_{(Z_{n-1}^{\vee}U_n')^{\omega_n}\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W\left(\hat{c}_{n,\ell}\overline{x}j_{n,\ell}(\delta_5 h)a_{n,\ell}\right)\widetilde{F_{\tilde{\phi}}}\left(m(w_{1,n}^{-1})h^{\omega_n}, I_n, I_{n-1}\right)d\overline{x}dh.$$

Here

$$\delta_5 = \begin{pmatrix} 0 & & & & & 1 \\ & -I_{n-2} & & & & \\ & & 0 & -1 & & \\ & & -1 & 0 & & \\ & & & & -I_{n-2} & \\ 1 & & & & & 0 \end{pmatrix}.$$

Change $h \mapsto \delta_5 h$ in (5.17) to get

$$\int_{Z^\vee_{n-1}U_n\backslash H_n}\int_{\overline{x}(n,\ell)} W\Big(\hat{c}_{n,\ell}\overline{x}j_{n,\ell}(h)a_{n,\ell}\Big)\widetilde{F_{\tilde{\phi}}}\Big(m(w_{1,n}^{-1})\delta_5 h^{\omega_n}, I_n, I_{n-1}\Big)d\overline{x}dh$$

$$= \int_{Z^\vee_{n-1}U_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W\Big(\hat{c}_{n,\ell}\overline{x}j_{n,\ell}(h)a_{n,\ell}\Big)$$

$$F_{M(\phi)}\Big(m\begin{pmatrix} & -I_{n-1} \\ 1 & \end{pmatrix}\cdot h\Big)^{\omega_n}, I_n, b^*_{n-1}d\overline{x}dh = \int_{Z^\vee_{n-1}U_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}}$$

$$W\Big(\hat{c}_{n,\ell}\overline{x}j_{n,\ell}(h)a_{n,\ell}\Big)F_{M(\phi)}\Big(h^{\omega_n}, \begin{pmatrix} 1 & \\ & -b^*_{n-1} \end{pmatrix} w_{1,n}, I_{n-1}\Big)d\overline{x}dh$$

$$= \int_{V_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W\Big(\hat{c}_{n,\ell}\overline{x}j_{n,\ell}(h)a_{n,\ell}\Big)\int_{F^{n-1}} F_{M(\phi)}\Bigg(h^{\omega_n}\begin{pmatrix} 1 & \\ & -b^*_{n-1} \end{pmatrix}$$

$$w_{1,n}\begin{pmatrix} I_{n-1} & z \\ 0 & 1 \end{pmatrix}, I_{n-1}\Bigg)\cdot\psi(z_{n-1})dzd\overline{x}dh$$

$$= \int_{V_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W\Big(\hat{c}_{n,\ell}\overline{x}j_{n,\ell}(h)a_{n,\ell}\Big)\int_{F^{n-1}}$$

$$F_{M(\phi)}\Bigg(h^{\omega_n}, w_{1,n}\begin{pmatrix} I_{n-1} & z \\ 0 & 1 \end{pmatrix} b^*_n, I_{n-1}\Bigg)\cdot\psi(z_{n-1})dzd\overline{x}dh$$

$$= \widetilde{A}(W, \varphi_\phi).$$

Assume that $n-1$ is odd. Change in (5.15) $h \mapsto \delta_3 h$. Note that $\delta_2\delta_3^\omega = \delta_5$. We get

$$\int_{(\overline{Y}V^n_n)^{\delta_3}\backslash H_n}\int_{\overline{X}_{(n-1,\ell-1)}}\int_{\overline{X}_{(1,\ell)}} W\Bigg(\overline{x}_1\begin{pmatrix} 1 & & \\ & \overline{x}_2 & \\ & & 1 \end{pmatrix} j_{n,\ell}(h)\Bigg)$$

$$\widetilde{F_{\tilde{\phi}}}(m(w_n^{-1})\delta_5 h^{\omega_n}, I_n, I_{n-1})d\overline{x}_1 d\overline{x}_2 dh. \tag{5.18}$$

Write $\overline{x}_1 \cdot \begin{pmatrix} 1 & & \\ & \overline{x}_2 & \\ & & 1 \end{pmatrix}$ in the form $\overline{x}\cdot a$, where $\overline{x}\in \overline{X}_{(n,\ell)}$ and

$$a = j_{n,\ell}\left(\begin{pmatrix} 1 & 0 & a_1 & 0 & a_2 & 0 & a_3 & 0 \\ & I_{n-3} & 0 & 0 & 0 & 0 & 0 & a_3' \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 & -a_2 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & -a_1 \\ & & & & & & I_{n-3} & 0 \\ & & & & & & & 1 \end{pmatrix}^{\delta_3}\right).$$

Note that $h \mapsto \widetilde{F_{\bar\phi}}(m(w_{1,n}^{-1})\delta_4 h^{\omega_n}, I_n, I_{n-1})$ is left invariant under a. We get

$$\int_{Z_{n-1}^{\vee}\widetilde{U}_{n-1}\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}j_{n,\ell}(h))\widetilde{F_{\bar\phi}}(m(w_{1,n}^{-1})\delta_5 h^{\omega_n}, I_n, I_{n-1})d\overline{x}dh$$

(where $\widetilde{U}_{n-1} = \left\{\begin{pmatrix} 1 & & \\ & u & \\ & & 1\end{pmatrix} \middle| (u \in U_{n-1}\right\}$)

$$= \int_{Z_{n-1}^{\vee}\widetilde{U}_{n-1}\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}j_{n,\ell}(h))F_{M(\phi)}\left(\left(m(w_{1,n}^{-1})\delta_5\right)^{\omega_n} h, I_n, b_{n-1}^*\right)d\overline{x}dh$$

(note that $(m(w_{1,n}^{-1})\delta_5)^{\omega_n} = m\begin{pmatrix} & -I_{n-1} \\ 1 & \end{pmatrix}$)

$$= \int_{Z_{n-1}^{\vee}\widetilde{U}_{n-1}\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}j_{n,\ell}(h))F_{M(\phi)}\left(h, \begin{pmatrix} 1 & \\ & -b_{n-1}^*\end{pmatrix} w_{1,n}, I_{n-1}\right)d\overline{x}dh$$

$$= \int_{V_n\backslash H_n}\int_{\overline{X}_{(n,\ell)}} W(\overline{x}j_{n,\ell}(h))\int_{F^{n-1}} F_{M(\phi)}\left(h, w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix} b_n^*, I_{n-1}\right)$$
$$\psi(z_{n-1})dzd\overline{x}ddh = \widetilde{A}(W,\varphi_\phi).$$

This completes the proof of Theorem 2. ∎

6. Justifications

We bring here the technical justifications of the formal manipulations performed in Section 3 (absolute convergence of integrals in certain domains and special substitutions). Those needed for Section 4 are easy repetitions of those of Section 3 and those needed for Section 5 are similar in nature (and easier) so we omit them.

a. PROOF OF LEMMA 3.1. We have to show the absolute convergence of the integral (3.2) in a domain of the form (3.3).

Using the Iwasawa decomposition, it is enough to consider

$$\int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a}) \int_{\overline{X}^{(\ell,n)}} \int_{F_{n-1}} |\phi(m(w_{1,n}) \begin{pmatrix} I_{n-1} & z \\ & 1 \end{pmatrix} \overline{x} i_{\ell,n}(\hat{a}), I_{2n-2}, I_{n-1})| dz d\overline{x} da. \tag{6.1}$$

Here δ_ℓ is the modular function with respect to the Borel subgroup of G_ℓ. Conjugating $i_{\ell,n}(\hat{a})$ to the left, we get (changing variables in $\overline{x}$ and in z)

$$\int_{A_\ell} |W(\widetilde{a})|\delta_\ell^{-1}(\hat{a})| \det a|^{s'-\zeta'+(1-n+2\ell)/2} \int_{\overline{X}^{(\ell,n)}} \int_{F_{n-1}} |\phi(m(w_{1,n} \begin{pmatrix} I_{n-1} & z \\ & 1 \end{pmatrix}) \overline{x}, I_{2n-2}, \begin{pmatrix} a & \\ & I_r \end{pmatrix})| dz d\overline{x} da. \tag{6.2}$$

Here $s' = \mathrm{Re}(s)$ and $\zeta' = \mathrm{Re}(\zeta)$. It is enough to replace the $d\overline{x}$-integration over $\overline{X}^{(\ell,n)}$ by that over the full lower Siegel radical, and show convergence. Conjugating by $m(w_{1,n})$ and replace ϕ by its right $m(w_{1,n})$ translate, we now consider

$$\int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a})| \det a|^{s'-\zeta'+(1-n+2\ell)/2} \int_{\overline{U}_n \times F_{n-1}} \left|\phi\left(m\begin{pmatrix} 1 & \\ z & I_{n-1} \end{pmatrix} \overline{u}(x), I_{2n-2}, \begin{pmatrix} a & \\ & I_r \end{pmatrix}\right)\right| dz dx da. \tag{6.3}$$

Write the following Iwasawa decompositions:

$$\begin{pmatrix} 1 & \\ z & I_{n-1} \end{pmatrix} = \begin{pmatrix} c & \\ & b_z \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & * \\ & & \ddots & \\ & & & 1 \end{pmatrix} k_z,$$

$$\overline{u}(x) = v_x t_x k_x,$$

where $k_z \in \mathrm{GL}_n(\mathcal{O})$, $k_x \in H_n(\mathcal{O})$, $v_x \in V_n$ and

$$b_z = \mathrm{diag}(b_2, \ldots, b_{n-1}), \quad t_x = \mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}).$$

Denote

$$[x] = \max\{1, |x|\}, \quad [z] = \max\{1, |z|\},$$

where $|\cdot|$ denotes the sup-norm. We have seen in [S1], Sect. 11.15 that

$$(6.4)\qquad [x]^{-2j} \le \left|\frac{t_j}{t_{j+1}}\right| \le [x]^{2j}, \quad j = 1, \dots, n-1,$$

$$(6.5)\qquad [x]^{-n} \le |t_1 \cdot \dots \cdot t_n| \le [x]^{-1},$$

$$(6.6)\qquad |t_1|^{-1} = [x_1],$$

$$(6.7)\qquad |b_i \cdot \dots \cdot b_n| = \max\{1, |z_i|, \dots, |z_n|\}, \quad i \ge 2,$$

where

$$x = \begin{pmatrix} x_n \\ \vdots \\ x_2 \\ x_1 \end{pmatrix}, \quad z = \begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

Note that, from (6.7),

$$(6.8)\qquad [z]^{-1} \le |b_i| \le [z], \quad i \ge 2,$$

$$(6.9)\qquad |c| = |b_2 \cdot \dots \cdot b_n|^{-1} = [z]^{-1}.$$

By (6.9) and (2.4), the inner dz-integration of (6.3) equals

$$(6.10)\qquad \int_{F_{n-1}} \int_{U_n} [z]^{-\mu'-2\zeta'-n/2} |\phi(\overline{u}(x)m(k_z), I_{2n-2}, \begin{pmatrix} a & \\ & I_r \end{pmatrix} b_z)| dz dx$$

where $|\mu(t)| = |t|^{\mu'}$. By (2.4), (6.5), (6.6), the integral (6.10) is majorized by

$$\int_{F_{n-1}} \int_{U_n} ([z][x_1])^{-\mu'-2\zeta'-n/2} [x]^{-N_0(\xi'-\zeta'+(n-3)/2)}$$

$$(6.11)\qquad |\phi\left(k_x m(k_z), I_{2n-2}, \begin{pmatrix} a & \\ & I_r \end{pmatrix} b_z t_x\right)| dz dx.$$

Here $N_0 = 1$ if $s' - \zeta' + (n-3)/2 \ge 0$, and $N_0 = n$ if $s' - \zeta' + (n-3)/2 < 0$. As in [S1], Sect. 4.4, we may majorize $|\phi(k_x m(k_z), I_{2n-2}, t)|$ by a linear combination with positive coefficients of positive quasi-characters $\eta(t)$. Here t is diagonal. By (6.4), (6.6), (6.8), we may consider, instead of (6.11),

$$(6.12)\qquad \eta(a) \int ([z] \cdot [x_1])^{-\mu'-2\zeta'-n/2} [x]^{-N_0(s'-\zeta'+(n-3)/2)+N} [z]^M dz dx$$

where M, N are positive and depend on τ' only. For the dz-integration to converge in (6.12), we must have

$$(6.13)\qquad -\mu' - 2\zeta' - \frac{1}{2}n + M < -M'$$

where M' is large enough. In particular, $-\mu' - 2\zeta' - \frac{1}{2}n < 0$, and since $[x_1] \geq 1$, (6.12) is majorized by a constant multiple of

$$\eta(a) \int [x]^{-N_0(s'-\zeta'+(n-3)/2)+N} dx.$$

For the dx-integration to converge, we must have

$$-N_0\Big(s' - \zeta' + \frac{n-3}{2}\Big) + N < -N' \tag{6.14}$$

where N' is large enough. Returning to (6.3), it remains to consider

$$\int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a})\eta(a)| \det a|^{s'-\xi'+(1-n+2\ell)/2} da.$$

Using the estimate in [S1], Sect. 2.3, the last integral converges in a domain of the form

$$s' - \zeta' > N'' \tag{6.15}$$

where N'' depends on π and η. Gathering the conditions (6.13)–(6.15) gives a domain of the form (3.3), and this concludes the proof of Lemma 3.1. ∎

b. Proof of Lemma 3.3. It is clear that it is enough to establish the absolute convergence of (3.8) in the domain (3.7). Using the Iwasawa decomposition, it is enough to consider

$$\begin{aligned} \int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a}) \int_{\overline{X}^{(\ell,n)}} \int_{F_{n-1}} |\widetilde{\phi}\Big(m\Big(w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1\end{pmatrix}\Big) \\ \overline{x}\delta_{\ell,n} i_{\ell,n}(\hat{a}), I_{2n-2}, I_{n-1}\Big)| d(z,\overline{x},a). \end{aligned} \tag{6.16}$$

Since $\delta_{\ell,n}$ and $i_{\ell,n}(\hat{a})$ commute, we may replace $\widetilde{\phi}$ by its right $\delta_{\ell,n}$ translate, and then consider (6.16) with $\delta_{\ell,n}$ omitted. Now (6.16) looks exactly like (6.1), with ϕ replaced by $\widetilde{\phi}$. Note that $\widetilde{\phi}$ lies in the space of $\mathrm{Ind}_{R_1}^{H_n}(\mu_{s+\zeta} \otimes \rho_{\tau',-(s-\zeta)})$. Now we repeat word for word the proof in 6.a, and get that the integral (3.6) converges absolutely in a domain of the form (3.7). ∎

c. Proof of Lemma 3.4. We have to compute $A(W, \varphi_\phi)$ and $B(W, \varphi_\phi)$ for special substitutions. Let ϕ_0 have support in R_1V, where V is a small neighbourhood of I_{2n}. By this we mean that

$$\phi_0(h, h', b) = 0, \quad h \notin R_1V,\ h' \in H_{n-1},\ b \in \mathrm{GL}_{n-1}(F). \tag{6.17}$$

Assume also that ϕ_0 is constant on V. Thus

$$\phi_0\left(\begin{pmatrix} x & * & * \\ & h_0 & * \\ & & x^{-1}\end{pmatrix} v, h', b\right) = \mu(x)|x|^{s+\zeta+n-\frac{3}{2}}\phi_0(I_{2n}, h'h_0', b), \quad v \in V$$

$$= \mu(x)|x|^{s+\zeta+n-\frac{3}{2}}\phi'(h'h_0', b), \tag{6.18}$$

where ϕ' lies in the space of $\rho_{\tau',s-\zeta} = \mathrm{Ind}_{Q_{n-1}}^{H_{n-1}(F)} \tau'_{s-\zeta}$. It follows from (6.17), (6.18) that

$$\phi_0^{\sim}(h, h', b) = 0, \quad h \notin R_1 V, \tag{6.19}$$

$$\phi_0^{\sim}\left(\begin{pmatrix} x & * & * \\ & h_0' & * \\ & & x^{-1}\end{pmatrix} v, h', b\right) = \mu(x)|x|^{s+\zeta+n-\frac{3}{2}}$$

$$M(w_{n-1}, \phi')((h'h_0')^{\omega_{n-1}^{n-1}}, b^*_{\ell,n-1}b^*), \quad v \in V. \tag{6.20}$$

Now let us take ϕ such that its right translate by $m(w_{1,n})\beta_{\ell,n}$ is ϕ_0, and use (3.4) to compute $A(W, \varphi_\phi)$. Choose V of the form $H_n \cap (I_{2n} + M_{2n}(\mathcal{P}^N))$ (N large enough). Then $\begin{pmatrix} 1 & & \\ v & I_{2n-2} & \\ * & v' & 1\end{pmatrix} \in R_1 V$ is equivalent to $\begin{pmatrix} 1 & & \\ v & I_{2n-2} & \\ * & v' & 1\end{pmatrix} \in V$. This shows that

$$A(W, \varphi_\phi) = \alpha \int_{N_\ell \backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n-1)}} \phi'(\overline{x} i_{\ell,n-1}(g), I_{n-1})\psi_a(\overline{x}) d\overline{x} dg \tag{6.21}$$

where α is the measure of the intersection of V and the unipotent radical of $\overline{R}_1$. Similarly, by (6.19), (6.20), we compute $B(W, \varphi_\phi)$ from (3.6) and get

$$B(W, \varphi_\phi) = \alpha \int_{N_\ell \backslash G_\ell} W(g) \int_{\overline{X}^{(\ell,n-1)}}$$

$$M(w_{n-1}, \phi')\left(\left(\overline{x}\alpha_{\ell,n-1} i_{\ell,n-1}(g)\right)^{\omega_{n-1}^{n-1}}, b^*_{\ell,n-1}\right)\psi_a^{(-1)^{n-1}}(\overline{x}) d\overline{x} dg. \tag{6.22}$$

The integrals (6.22) and (6.21) (as meromorphic functions) are proportional by the factor $\frac{\gamma(\pi\times\tau', s-\zeta, \psi)}{\gamma(\tau', \Lambda^2, 2(s-\zeta)-1, \psi)}$, by the local functional equation for π and τ on $G_\ell \times \mathrm{GL}_{n-1}(F)$. ∎

d. Proof of Lemma 3.6. Note that

$$\widetilde{F}_{\tilde{\phi}}\left(\begin{pmatrix} m_0 & \\ & m_0^* \end{pmatrix} h, m, r\right) = \widetilde{F_{\tilde{\phi}}}(h, mm_0, r), \tag{6.23}$$

(6.24)
$$\widetilde{F_{\tilde\phi}}\left(h, \begin{pmatrix} b & * \\ & x \end{pmatrix} m, r\right) = \mu(x)|x|^{s+\zeta-(n+1)/2}|\det b|^{-s+\zeta+(n+1)/2}\widetilde{F_{\tilde\phi}}(h,m,rb),$$

for $b \in \mathrm{GL}_{n-1}(F)$, $x \in F^*$. Using the Iwasawa decomposition in (3.12), it is enough to take $g = \hat{a} \in \hat{A}_\ell$ and omit $\delta_{\ell,n}$. Conjugating $\overline{x} \mapsto i_{\ell,n}(\hat{a})\overline{x} i_{\ell,n}(\hat{a}^{-1})$, using (6.23), (6.24), and changing variable in t (in (3.12)), we obtain

$$\int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a})|a_1|^{2s+\mu'-n-1}|\det a|^{-s+\zeta+(2\ell+1-n)/2}$$
$$\int_{\overline{X}^{(\ell,n)}} \int_{F^{n-1}} |\widetilde{F_{\tilde\phi}}\left(\overline{x}, w_{1,n}^{-1}\begin{pmatrix} 1 & t \\ & I_{n-1}\end{pmatrix}\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1}\end{pmatrix},\right.$$
$$\left.\begin{pmatrix} a_2 & & & \\ & \ddots & & \\ & & a_\ell & \\ & & & I_{r+1}\end{pmatrix}\right)\Bigg|d(t,\overline{x},a). \tag{6.25}$$

We will determine convergence of the integral obtained from (6.25) by replacing $\overline{X}^{(\ell,n)}$ with the full radical $\overline{U}_n$. (This, of course, will imply convergence of (6.25).) Thus (after simple conjugations, and replacing $\widetilde{F_{\tilde\phi}}$ by a translate by a Weyl element), we may consider

$$\int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a})|a_1|^{2s'+\mu'-n-1}|\det a|^{-s'+\zeta'+(2\ell+1-n)/2}\int_{\overline{U}_n\times F^{n-1}}$$
$$|\widetilde{F_{\tilde\phi}}\left(\overline{u}(x), \begin{pmatrix} I_{n-1} & \\ z & 1\end{pmatrix}\begin{pmatrix} a_2 & & & \\ & \ddots & & \\ & & a_\ell & \\ & & & I_{r+1}\end{pmatrix}\right)\Bigg|d(z,x,a). \tag{6.26}$$

Write the Iwasawa decomposition

$$\begin{pmatrix} I_{n-1} & \\ z & 1\end{pmatrix} = \begin{pmatrix} c_z & 0 \\ 0 & e\end{pmatrix}\begin{pmatrix} 1 & & & * \\ & \ddots & & \\ & & 1 & \\ & & & 1\end{pmatrix} k_z$$

where $c_z = \mathrm{diag}(c_1,\ldots,c_{n-1})$ and $k_z \in \mathrm{GL}_n(\mathcal{O})$. Note that

$$[z] = \max\{1, |\,z|\,\} = |e| = |\det c_z|^{-1}.$$

In general,

$$|c_i c_{i+1}\cdot\ldots\cdot c_{n-1} e| = \max\{1, |z_1|, |z_2|, \ldots, |z_{i-1}\}$$

and hence

$$[z]^{-1} \le |c_i| = \frac{\max\{1, |z_1|, \ldots, |z_{i-1}\}}{\max\{1, |z_1|, \ldots, |z_i|\}} \le [z]. \tag{6.27}$$

Using (6.23), (6.24), and conjugating $\overline{u}(x)$ by $m(k_z)$, (6.26) becomes

$$\int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a})|a_1|^{2s'+\mu'-n-1}|\det a|^{-s'+\zeta'+(2\ell+1-n)/2}$$
$$\int_{\overline{U}_n \times F^{n-1}} [z]^{2s'+\mu'-n-1}\Big|\widetilde{F_{\tilde{\phi}}}(\overline{u}(x)m(k_z), I_n,$$
$$\begin{pmatrix} a_2 & & & \\ & \ddots & & \\ & & a_\ell & \\ & & & I_{r+1} \end{pmatrix} c_z)\Big| d(z,x,a). \tag{6.28}$$

Now write the Iwasawa decomposition of $\overline{u}(x)$ as in 6.1 (with the same notation) and recall (6.4)–(6.6). Using (6.23), (6.24), we see, as in 6.a, that it suffices to consider instead of (6.28)

$$\int_{A_\ell} |W(\hat{a})|\delta_\ell^{-1}(\hat{a})|a_1|^{2s'+\mu'-n-1}|\det a|^{-s'+\zeta'+(2\ell+1-n)/2}$$
$$\int [z]^{2s'+\mu'+n-1}|\det(t)|^{s'+\zeta'+\mu'-(n+1)/2} \cdot |t_1 \cdot \ldots \cdot t_{n-1}|^{-2s'+\mu'+n+1}|$$
$$\widetilde{F_{\tilde{\phi}}}\left(I_{2n}, I_n, \begin{pmatrix} a_2 & & & \\ & \ddots & & \\ & & a_\ell & \\ & & & I_{r+1} \end{pmatrix} c_z t_x\right)\Big| d(z,x,a). \tag{6.29}$$

Now majorize $|\widetilde{F_{\tilde{\phi}}}(I_{2n}, I_n, r)|$ by a gauge on $GL_{n-1}(F)$ (see [S1], Sect. 2.3). Thus, for the dz-integration in (6.29), we have to require

$$s' < -M_1 \tag{6.30}$$

where $M_1 \gg 0$ (depending on τ' and μ). We may take M_1 large enough, so that for s as in (6.30), $-2s'-\mu'+n+1 > 0$. It is easy to see that $|t_1, \ldots, t_{n-1}| \le 1$, and hence $|t_1 \cdot \ldots \cdot t_{n-1}|^{-2s'-\mu'+n+1} \le 1$. The da-integrations will require conditions of the form

$$s' + \zeta' > M_2 \tag{6.31}$$

and

$$-s' + \zeta' > M_3$$

where $M_2, M_3 \gg 0$ and depend on π, μ, τ'. We may take M_2 so large that $s' + \zeta' + \mu' - (n+1)/2 > 0$, and then, since $|\det(t)| \leq [x]^{-1}$, the dt-integration is majorized by

$$\int [x]^{-(s'+\zeta'+\mu'-(n+1)/2)} dx$$

which converges due to (6.31). The conditions (6.29)-(6.31) give a domain of the form (3.14). ∎

e. Proof of Lemma 3.7. We have to compute the integrals in both sides of (3.12) for a special substitution as we did in Sect. 6.c. We make the same substitutions as we did in [S1], Prop. 6.2 (for W and for $F_{\tilde{\phi}}$ replacing $\xi_{\tau,s}$). We get that $B(W, \varphi_\phi)$ (we use the form (3.9)) equals

$$c \int_{F_{n-1}} F_{\tilde{\phi}}\left(I_{2n}, w_{1,n}\begin{pmatrix} I_{n-1} & z \\ & 1 \end{pmatrix}, I_{n-1}\right)\psi(z_{n-1})dz \tag{6.32}$$

(the constant c is a measure of a unipotent group close to I_{2n}). The same substitution to the r.h.s. of (3.12) gives
(6.33)

$$c \int_{F^{n-1}} \widetilde{F_{\tilde{\phi}}}\left(I_{2n}, w_{1,n}^{-1}\begin{pmatrix} 1 & t \\ & I_{n-1} \end{pmatrix}\begin{pmatrix} I_{\ell+1} & & \\ & (-1)^{n-1} & \\ & & I_{r-1} \end{pmatrix}, I_{n-1}\right)\psi(t_1)dt$$

(with the same constant c). The proportionality factor between (6.32) and (6.33) is the local coefficient $c_\psi(\mu_{s+\zeta+(n-1)/2} \times (\tau')^*_{-(s-\zeta)+n/2})$. ∎

f. Proof of Lemma 3.10. Since all manipulations in the proof of Lemma 3.11 and those leading to (3.33) and (3.34) are formal, i.e. consist of variable changes and integration collapsing, it is enough to establish a domain of absolute convergence of (3.33) and (3.34), which define $\widetilde{A}(W, \varphi_\phi)$. Thus it remains to apply Lemma 3.1, with $(-\zeta, -s)$ replacing (ζ, s). ∎

References

[J.PS.S.] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Rankin–Selberg convolutions*, American Journal of Mathematics **105** (1983), 367–483.

[Sh1] F. Shahidi, *A proof of Langlands conjecture on Plancherel measures: complementary series for p-adic groups*, Annals of Mathematics **132** (1990), 273–330.

[Sh2] F. Shahidi, *On certain L-functions*, American Journal of Mathematics **103** (1981), 297–355.

[Sh3] F. Shahidi, *Fourier transforms of intertwining operators and Plancherel measures for* GL(n), American Journal of Mathematics **106** (1984), 67–111.

[S1] D. Soudry, *Rankin–Selberg convolutions for* $SO_{2\ell+1} \times GL_n$*: Local theory*, Memoirs of the American Mathematical Society **500** (1993), 1–100.

[S2] D. Soudry, *On the archimedean theory of Rankin–Selberg convolutions for* $SO_{2\ell+1} \times GL_n$, Annales Scientifiques de l'École Normale Supérieure **28** (1995), 161–224.